

Exchange of three intervals: itineraries, substitutions and palindromicity

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Abstract

Given a symmetric exchange of three intervals, we provide a detailed description of the return times to a subinterval and the corresponding itineraries. We apply our results to morphisms fixing words coding non-degenerate three interval exchange transformation. This allows us to prove that the conjecture stated by Hof, Knill and Simon is valid for such infinite words.

1 Introduction

Interval exchange transformations have been extensively studied since the works on their ergodic aspects by Sinai [27], Keane [14], Veech [31], Rauzy [24], and others. For an overview, see [30] and references therein. Among general dynamical systems, interval exchanges have the interesting property that the Poincaré first return map is again a mapping of the same type, i.e. an exchange of (possibly different number of) intervals. Rauzy [23] used this fact to present a generalization of the classical continued fractions expansion.

It is commonly known that interval exchange transformations provide a very useful framework for the study of infinite words arising by coding of rotations, in particular Sturmian words. These are usually defined as aperiodic infinite words with lowest factor complexity. Equivalently, one obtains Sturmian words by binary coding of the trajectory under exchange T of two intervals $[0, \alpha)$, $[\alpha, 1)$ with α irrational. Given a subinterval $I \subset [0, 1)$, the first return map T_I to I is an exchange of at most three intervals, although the return itineraries of points can take up to four values. The set of these itineraries can be used to describe certain characteristics of Sturmian words, namely the return words, see [32], or abelian return words [25], and invariance under morphisms [34].

Infinite words coding exchange of k -intervals, $k \geq 3$, are also in focus for several decades [10, 11] and, here too, one finds a close relation between their combinatorial features and the properties of the induced map, see for example [33] for a result on return words or [1] about substitutivity of interval exchange words. A generalized version of the Poincaré first return map was used in [9] for description of palindromic complexity in codings of rotations. These words are in intimate relation with three interval exchange words.

In this paper we focus on codings of a non-degenerate symmetric exchange $T : J \rightarrow J$ of three intervals. First we describe the return times to a general interval $I \subset J$ and provide an insight on the structure of the set of I -itineraries. These results are given as Theorem 5.1 and then interpreted as analogues of the well known three gap and three distance theorems.

A particular attention is paid to the special cases when the set of I -itineraries has only three elements. These cases belong to the most interesting from the combinatorial point of view, since

they provide information about return words to factors, and about the morphisms preserving three interval exchange words. For mutually conjugated morphisms, we describe in Theorem 9.10 the relation between intercepts of their fixed points, as was done for Sturmian morphisms in [21]. We also show that morphisms conjugated only to themselves do not have a non-degenerated fixed point.

The most important application of our results is a contribution to the solution of the question stated by Hof, Knill and Simon [13] for palindromic words. We refer to it as the HKS conjecture and adopt its reformulation by Tan [29] who showed its validity for binary words. Labbé [16] presented a counterexample for the conjecture on ternary alphabet; the ternary word not satisfying the hypothesis turns out to be a degenerate three interval exchange word. In fact, degenerate three interval exchange words are just morphic images of binary, in fact Sturmian, words. In this paper we show that for non-degenerated words coding exchange of three intervals the HKS conjecture holds. Let us mention that the latter result has been announced at the DLT conference [18]. Here we provide a full proof.

This paper is organized as follows. Section 2 contains the necessary notions from combinatorics on words. Symmetric k -interval exchange transformations and their properties with respect to the first return map are treated in Section 3. Section 4 focuses on specific properties when $k = 3$. The main theorem about return times in three interval exchanges is given in Section 5. In Section 6 we put our results into context of three gaps and three distance theorems.

The specific case when the set of I -itineraries has only three elements is studied in Section 7. This allows us to describe the return words to palindromic bispecial factors. In Section 8 we focus on substitution invariance of words coding interval exchange transformations. The key lemma for the demonstration of our Theorem 10.3 on HKS conjecture requires some knowledge about the relation of substitutions fixing words coding three interval exchange and Sturmian morphisms. This topic is treated in Section 9. The proof of Theorem 10.3 is then provided in Section 10 together with some other comments.

2 Preliminaries

Let us recall necessary notions and notation from combinatorics on words. For a basic overview we refer to [17]. An *alphabet* is a finite set of symbols, called *letters*. A *finite word* w over an alphabet \mathcal{A} of length $|w| = n$ is a concatenation $w = w_0 \cdots w_{n-1}$ of letters $w_i \in \mathcal{A}$. The set of all finite words over \mathcal{A} equipped with the operation of concatenation and the empty word ϵ is a monoid denoted by \mathcal{A}^* . For a fixed letter $a \in \mathcal{A}$, the number of occurrences of a in w , i.e., the number of indices i such that $w_i = a$, is denoted by $|w|_a$. The *reversal* or *mirror image* of the word w is the word $\overline{w} = w_{n-1} \cdots w_0$. A word w for which $w = \overline{w}$ is called a *palindrome*. An *infinite word* \mathbf{u} is an infinite concatenation $\mathbf{u} = u_0 u_1 u_2 \dots \in \mathcal{A}^{\mathbb{N}}$. An infinite word $\mathbf{u} = wvvv \dots$ with $w, v \in \mathcal{A}^*$ is said to be *eventually periodic*; it is said to be *aperiodic* if it is not of such form. We say that $w \in \mathcal{A}^*$ is a *factor* of $v \in \mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}}$ if $v = w'wv''$ for some $w' \in \mathcal{A}^*$ and $v'' \in \mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}}$. If $w' = \epsilon$ or $v'' = \epsilon$, then w is a *prefix* or *suffix* of v , respectively. If $v = wu$, then we write $u = w^{-1}v$ and $w = vu^{-1}$.

The set $\mathcal{L}(\mathbf{u})$ of all finite factors of an infinite word \mathbf{u} is called the *language of \mathbf{u}* . If for any factor $w \in \mathcal{L}(\mathbf{u})$ there exist at least two indices i such that w is a prefix of the infinite word $u_i u_{i+1} u_{i+2} \dots$, the word \mathbf{u} is *recurrent*. Given a factor $w \in \mathcal{L}(\mathbf{u})$, a finite word v such that vw belongs to $\mathcal{L}(\mathbf{u})$ and the word w occurs in vw exactly twice, once as a prefix and once as a suffix of vw , is called a *return word of w* . If any factor w of an infinite recurrent word \mathbf{u} has only finitely many return words, the word \mathbf{u} is called *uniformly recurrent*.

The *factor complexity* $\mathcal{C}_{\mathbf{u}}$ is the function $\mathbb{N} \rightarrow \mathbb{N}$ counting the number of factors of \mathbf{u} of length n . It is known that the factor complexity of an aperiodic infinite word \mathbf{u} satisfies $\mathcal{C}_{\mathbf{u}}(n) \geq n + 1$ for all n . Aperiodic infinite words having the minimal complexity $\mathcal{C}_{\mathbf{u}}(n) = n + 1$ for all n are called *Sturmian words*. Since $\mathcal{C}_{\mathbf{u}}(1) = 2$, they are binary words. Sturmian words can be equivalently defined in many different frameworks, one of them is coding of an exchange of two intervals.

Let \mathcal{A} and \mathcal{B} be alphabets. Let $\varphi : \mathcal{A}^* \rightarrow \mathcal{B}^*$ be a *morphism*, i.e., $\varphi(wv) = \varphi(w)\varphi(v)$ for all $w, v \in \mathcal{A}^*$. We say that φ is *non-erasing* if $\varphi(b) \neq \epsilon$ for every $b \in \mathcal{A}$. The action of φ can be naturally extended to infinite words $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ by setting $\varphi(\mathbf{u}) = \varphi(u_0)\varphi(u_1)\varphi(u_2)\dots$. If $\mathcal{A} = \mathcal{B}$ and $\varphi(\mathbf{u}) = \mathbf{u}$, then \mathbf{u} is said to be a *fixed point* of φ . A non-erasing morphism $\varphi : \mathcal{A}^* \rightarrow \mathcal{A}^*$ such that there is a letter $a \in \mathcal{A}$ satisfying $\varphi(a) = aw$ for some non-empty word w is called a *substitution*. Obviously, a substitution has always a fixed point, namely $\lim_{n \rightarrow \infty} \varphi^n(a)$ where the limit is taken over the product topology. An infinite word which is a fixed point of a substitution is called a *pure morphic* word. Let $\mathcal{A} = \{a_1, \dots, a_k\}$ and $\mathcal{B} = \{b_1, \dots, b_\ell\}$. One associates to every morphism $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ its *incidence matrix* $M_\varphi \in \mathbb{N}^{k \times \ell}$ defined by

$$(M_\varphi)_{ij} = |\varphi(a_i)|_{b_j}, \quad \text{for } 1 \leq i \leq k, 1 \leq j \leq \ell.$$

A morphism $\varphi : \mathcal{A}^* \rightarrow \mathcal{A}^*$ is said to be *primitive* if all elements of some power of its incidence matrix $M_\varphi \in \mathbb{N}^{k \times k}$ are positive. A specific class of morphisms is formed by the so-called *Sturmian morphisms* which are defined over the binary alphabet $\{0, 1\}$ and for which there exists a Sturmian word \mathbf{u} such that $\varphi(\mathbf{u})$ is also Sturmian. For an overview about properties of Sturmian morphisms see [17].

3 Itineraries in symmetric exchange of intervals

For disjoint intervals K and K' we write $K < K'$ if for $x \in K$ and $x' \in K'$ we have $x < x'$. Let J be a semi-closed interval. Consider a partition $J = J_0 \cup \dots \cup J_{k-1}$ of J into a disjoint union of semi-closed subintervals $J_0 < J_1 < \dots < J_{k-1}$. A bijection $T : J \rightarrow J$ is called an *exchange of k intervals with permutation π* if there exist numbers c_0, \dots, c_{k-1} such that for $0 \leq i < k$ one has

$$T(x) = x + c_i \text{ for } x \in J_i, \quad (1)$$

where π is a permutation of $\{0, 1, \dots, k-1\}$ such that $T(J_i) < T(J_j)$ for $\pi(i) < \pi(j)$. In other words, the permutation π determines the order of intervals $T(J_i)$. If π is the permutation $i \mapsto k - i + 1$, then T is called a *symmetric* interval exchange.

The orbit of a given point ρ is the infinite sequence $\rho, T(\rho), T^2(\rho), T^3(\rho), \dots$. It can be coded by an infinite word $\mathbf{u}_\rho = u_0 u_1 u_2 \dots$ over the alphabet $\{0, 1, \dots, k-1\}$ given by

$$u_n = X \quad \text{if } T^n(\rho) \in J_X \quad \text{for } X \in \{0, 1, \dots, k-1\}.$$

The point ρ is called the *intercept* of \mathbf{u}_ρ . An exchange of intervals satisfies the *minimality condition* if the orbit of any given $\rho \in [0, 1)$ is dense in J . In this case, the word \mathbf{u}_ρ is aperiodic, uniformly recurrent, and the language of \mathbf{u}_ρ depends only on the parameters of the transformation T and not on the intercept ρ itself. The complexity of an infinite word \mathbf{u}_ρ is known to satisfy $\mathcal{C}_{\mathbf{u}_\rho}(n) \leq (k-1)n + 1$ (see [10]). If for every $n \in \mathbb{N}$ we have $\mathcal{C}_{\mathbf{u}_\rho}(n) = (k-1)n + 1$, then the transformation T and the word \mathbf{u}_ρ are said to be *non-degenerate*. A sufficient and necessary condition on T to be non-degenerate is that the orbits of the discontinuity points of T are infinite and disjoint. This condition is known under the abbreviation i.d.o.c.

Definition 3.1. Let T be an exchange of k intervals satisfying the minimality condition. Given a subinterval $I \subset J$, we define the mapping $r_I : I \rightarrow \mathbb{Z}^+ = \{1, 2, 3, \dots\}$ by

$$r_I(x) = \min\{n \in \mathbb{Z}^+ : T^n(x) \in I\},$$

the so-called *return time* to I . The prefix of length $r_I(x)$ of the word \mathbf{u}_x coding the orbit of a point $x \in I$ is called the *I -itinerary* of x and denoted $R_I(x)$. The set of all I -itineraries is denoted by $\text{It}_I = \{R_I(x) : x \in I\}$. The mapping $T_I : I \rightarrow I$ defined by

$$T_I(x) = T^{r_I(x)}(x)$$

is said to be the *first return map* of T to I , or *induced map* of T on I .

Throughout the paper, when it is clear from the context, we sometimes omit the index I in r_I or R_I . It is known from Keane [14] that if T is an exchange of k intervals and $I \subset J$, then It_I has at most $k + 2$ elements, and, consequently, T_I is an exchange of at most $k + 2$ intervals.

Remark 3.2. Let $X \in \{0, 1, \dots, k - 1\}$. If $I \subset J_X$, then $T(I)$ is an interval and we have

$$R \text{ is an } I\text{-itinerary} \Leftrightarrow X^{-1}RX \text{ is a } T(I)\text{-itinerary}.$$

Similarly, if $I \subset T(J_X)$, then $T^{-1}(I)$ is an interval and we have

$$R \text{ is an } I\text{-itinerary} \Leftrightarrow XRX^{-1} \text{ is a } T^{-1}(I)\text{-itinerary}.$$

We will use another fact about itineraries of an interval exchange. Without loss of generality, we consider $J = [0, 1)$. The intervals J_X are left-closed right-open for all $X \in \{0, 1, \dots, k - 1\}$. Such interval exchange T is right-continuous. Therefore, if $I = [\gamma, \delta)$, then every word $w \in \text{It}_I = \{R(x) : x \in I\}$ is an I -itinerary $R(x)$ for infinitely many $x \in I$, which form an interval, again left-closed right-open.

Proposition 3.3. *Let T be a k -interval exchange satisfying the minimality condition and let $I = [\gamma, \delta) \subset [0, 1)$. There exist neighbourhoods H_γ and H_δ of γ and δ , respectively, such that for every $\tilde{\gamma} \in H_\gamma$ and $\tilde{\delta} \in H_\delta$ with $0 \leq \tilde{\gamma} < \tilde{\delta} \leq 1$ one has*

$$\text{It}_{\tilde{I}} \supseteq \text{It}_I, \quad \text{where } \tilde{I} = [\tilde{\gamma}, \tilde{\delta}).$$

In particular, if $\#\text{It}_I = k + 2$, then $\text{It}_{\tilde{I}} = \text{It}_I$.

Proof. Let $\text{It}_I = \{R_1, \dots, R_m\}$, $m \in \mathbb{N}$, and $I_i = \{x \in I : R_I(x) = R_i\}$ for $1 \leq i \leq m$. As already mentioned, $m \leq k + 2$. For every $i = 1, \dots, m$, consider arbitrary x_i in the interior of I_i . Such x_i satisfies that $T^j(x_i) \notin \{\gamma, \delta\}$ for $0 \leq j \leq r_I(x_i) = |R_I(x_i)|$. The reason is simple: if $T^j(x_i)$ were equal to γ (or δ), then a point x in I_i with $x < x_i$ (or $x > x_i$) would have a longer return time than x_i itself, which is a contradiction. Denote $M = \{T^j(x_i) : 0 \leq j \leq r_I(x_i), i = 1, \dots, m\}$ and

$$\varepsilon := \min\{|y - z| : y \in M, z \in \{\gamma, \delta\}\}.$$

Let $H_\gamma = (\gamma - \varepsilon, \gamma + \varepsilon)$ and $H_\delta = (\delta - \varepsilon, \delta + \varepsilon)$ be neighbourhoods of γ and δ , respectively. If $\tilde{\gamma} \in H_\gamma$ and $\tilde{\delta} \in H_\delta$ with $0 \leq \tilde{\gamma} < \tilde{\delta} \leq 1$ and $\tilde{I} = [\tilde{\gamma}, \tilde{\delta})$, then clearly for every $i = 1, \dots, m$ we have $x_i \in \tilde{I}$ and

$$T^j(x_i) \in I \Leftrightarrow T^j(x_i) \in \tilde{I} \quad \text{for } 0 \leq j \leq r_I(x_i).$$

Therefore the point x_i has the same return time with respect to I as to \tilde{I} . Consequently, the \tilde{I} -itinerary of x_i coincides with the I -itinerary of x_i . Thus $\text{It}_I \subseteq \text{It}_{\tilde{I}}$. \square

For the rest of the section, we consider only symmetric interval exchange. In order to state a property of such interval exchanges, for an interval $K = [c, d) \subset [0, 1)$ we set $\overline{K} = [1 - d, 1 - c)$. In this notation

$$T(J_X) = \overline{J_X} \text{ for any letter } X \in \{0, 1, \dots, k-1\}. \quad (2)$$

Proposition 3.4. *Let $T : [0, 1) \rightarrow [0, 1)$ be a symmetric exchange of k intervals satisfying the minimality condition. Let $I \subset [0, 1)$ and let R_1, \dots, R_m be the I -itineraries. The \overline{I} -itineraries are the mirror images of the I -itineraries, namely $\overline{R_1}, \dots, \overline{R_m}$. Moreover, if*

$$[\gamma_j, \delta_j) := \{x \in I : R_I(x) = R_j\} \quad \text{and} \quad [\gamma'_j, \delta'_j) := T_I[\gamma_j, \delta_j),$$

for $j = 1, \dots, m$, then

$$\{x \in \overline{I} : R_{\overline{I}}(x) = \overline{R_j}\} = [1 - \delta'_j, 1 - \gamma'_j).$$

Proof. Consider the restriction of the transformation T to the set

$$S = [0, 1) \setminus \{T^j(\alpha) : j \in \mathbb{Z}, \alpha \text{ is a discontinuity of } T\}.$$

Such a restriction is a bijection $S \rightarrow S$. We will show by induction that for any $i \geq 1$ and $y \in S$

$$T^{-i}(y) = 1 - T^i(1 - y). \quad (3)$$

Let $y \in S$ and $j \in \{0, \dots, k-1\}$ such that $y \in I_j$. Since T is symmetric, we have

$$1 - y \in I_j \Leftrightarrow y \in T(I_j).$$

The last equivalence and the definition of T imply

$$\begin{aligned} T(1 - y) &= 1 - y + c_j & \text{and} \\ T^{-1}(y) &= y - c_j. \end{aligned}$$

Summing the last two equalities we obtain

$$T^{-1}(y) = 1 - T(1 - y).$$

Then, using the induction hypothesis, we have for $y \in S$ that

$$T^{-(i+1)}(y) = T^{-1}(T^{-i}(y)) = 1 - T(1 - T^{-i}(y)) = 1 - T(T^i(1 - y)) = 1 - T^{i+1}(1 - y),$$

which proves (3).

Using (2) we can write for $y \in S$

$$T^{-1}(y) \in J_X \Leftrightarrow y \in T(J_X) \Leftrightarrow 1 - y \in J_X. \quad (4)$$

More generally,

$$T^{-i}(y) = T^{-1}(T^{-(i-1)}(y)) \in J_X \Leftrightarrow 1 - T^{-(i-1)}(y) = T^{i-1}(1 - y) \in J_X, \quad (5)$$

where we have first used (4) and then (3).

Now we show that if R_j is an I -itinerary, then its mirror image $\overline{R_j}$ is an \overline{I} -itinerary. Consider $\rho \in (\gamma_j, \delta_j) \cap S$ and let $R_I(\rho) = a_0 a_1 \dots a_{n-1}$ be its I -itinerary, i.e., $a_i = X$ if and only if $T^i(\rho) \in J_X$. Moreover, $T^i(\rho) \notin I$ for $1 \leq i < n$, and $T^n(\rho) \in I$. Let

$$\rho' := 1 - T^n(\rho) = 1 - T_I(\rho) \in (1 - \delta'_j, 1 - \gamma'_j) \cap S \subset \overline{I}. \quad (6)$$

By (3), we have $\rho' = T^{-n}(1-\rho)$, and therefore again by (3), $T^i(\rho') = T^{-(n-i)}(1-\rho) = 1 - T^{n-i}(\rho) \notin \bar{I}$ for $0 < i < n$. On the other hand, $T^n(\rho') = 1 - \rho \in \bar{I}$. By (5), we have for $i = 0, 1, \dots, n-1$ that

$$J_X \ni T^i(\rho') = T^{-(n-i)}(1-\rho) \Leftrightarrow T^{n-i-1}(\rho) \in J_X,$$

which implies that the \bar{I} -itinerary of ρ' is $R_{\bar{I}}(\rho') = a_{n-1}a_{n-2}\cdots a_0$, as we wanted to show.

By right continuity of T , all points from $[1 - \delta'_j, 1 - \gamma'_j)$ have the same \bar{I} -itinerary as $\rho' \in (1 - \delta'_j, 1 - \gamma'_j) \cap S$. \square

The above auxiliary statements will be used in Section 5 for the description of I -itineraries in exchanges of three intervals. Analogous result for exchange of two intervals was given in [19]. The claim of the last proposition also partially follows from the work done in [10].

4 Exchange of three intervals

We will be particularly interested in exchange of three intervals. For reasons that will appear later, we prefer to use for its coding the ternary alphabet $\{A, B, C\}$ instead of $\{0, 1, 2\}$. Without loss of generality let $0 < \alpha < \beta < 1$. Let $T : [0, 1) \rightarrow [0, 1)$ be given by

$$T(x) = \begin{cases} x + 1 - \alpha & \text{if } x \in [0, \alpha) =: J_A, \\ x + 1 - \alpha - \beta & \text{if } x \in [\alpha, \beta) =: J_B, \\ x - \beta & \text{if } x \in [\beta, 1) =: J_C. \end{cases} \quad (7)$$

The transformation T is an exchange of three intervals with the permutation (321). It is often called a *3iet* for short. The infinite word \mathbf{u}_ρ coding the orbit of a point $\rho \in [0, 1)$ under a 3iet is called a *3iet word*.

We require that $1 - \alpha$ and β be linearly independent over \mathbb{Q} , which is known to be a necessary and sufficient condition for minimality of the 3iet T . Non-degeneracy of T is equivalent to the condition of minimality together with

$$1 \notin (1 - \alpha)\mathbb{Z} + \beta\mathbb{Z}, \quad (8)$$

see [10]. This means that the 3iet word \mathbf{u} has complexity $\mathcal{C}_{\mathbf{u}}(n) = 2n + 1$ if and only if the parameters α and β of the corresponding 3iet T satisfy (8).

For a given subinterval $I \subset [0, 1)$ there exist at most five I -itineraries under a 3iet T . In particular, from the paper of Keane [14], one can deduce what are the intervals of points with the same itinerary. We summarize it as the following lemma.

Lemma 4.1. *Let T be a 3iet defined by (7) and let $I = [\gamma, \delta) \subset [0, 1)$ such that $\delta < 1$. Denote*

$$\begin{aligned} k_\alpha &:= \min \{k \in \mathbb{Z}, k \geq 0 : T^{-k}(\alpha) \in (\gamma, \delta)\}, \\ k_\beta &:= \min \{k \in \mathbb{Z}, k \geq 0 : T^{-k}(\beta) \in (\gamma, \delta)\}, \\ k_\gamma &:= \min \{k \in \mathbb{Z}, k \geq 1 : T^{-k}(\gamma) \in (\gamma, \delta)\}, \\ k_\delta &:= \min \{k \in \mathbb{Z}, k \geq 1 : T^{-k}(\delta) \in (\gamma, \delta)\}, \end{aligned}$$

and further

$$\mathfrak{A} := T^{-k_\alpha}(\alpha), \quad \mathfrak{B} := T^{-k_\beta}(\beta), \quad \mathfrak{C} := T^{-k_\gamma}(\gamma), \quad \mathfrak{D} := T^{-k_\delta}(\delta).$$

For $x \in I$, let K_x be a maximal interval such that for every $y \in K_x$, we have $R(y) = R(x)$. Then K_x is of the form $[c, d)$ with $c, d \in \{\gamma, \delta, \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}\}$. Consequently, $\#\text{It}_I \leq 5$.

For a 3iet T , Lemma 4.1 implies the already mentioned result that T_I is an exchange of at most 5 intervals. Consequently, the transformation T_I has at most four discontinuity points. In fact, the following result of [6] says that independently of the number of I -itineraries, the induced map T_I has always at most two discontinuity points.

Proposition 4.2 ([6]). *Let $T : J \rightarrow J$ be a 3iet with the permutation (321) and satisfying the minimality condition, and let $I \subset J$ be an interval. The first return map T_I is either a 3iet with permutation (321) or a 2iet with permutation (21). In particular, in the notation of Lemma 4.1, we have $\mathfrak{D} \leq \mathfrak{C}$.*

Convention: For the rest of the paper, let T be a non-degenerate exchange of three intervals with permutation (321) given by (7).

5 Return time in a 3iet

The aim of this section is to describe the possible return times of a non-degenerate 3iet T to a general subinterval $I \subset [0, 1)$. Our aim is to prove the following theorem.

Theorem 5.1. *Let T be a non-degenerate 3iet and let $I \subset [0, 1)$. There exist positive integers r_1, r_2 such that the return time of any $x \in I$ takes value in the set $\{r_1, r_1 + 1, r_2, r_1 + r_2, r_1 + r_2 + 1\}$ or $\{r_1, r_1 + 1, r_2, r_2 + 1, r_1 + r_2 + 1\}$.*

First, we will formulate an important lemma, which needs the following notation. Given letters $X, Y, Z \in \{A, B, C\}$ and a finite word $w \in \{A, B, C\}^*$, Let $\omega_{XY \rightarrow Z}(w)$ be the set of words obtained from w replacing one factor XY by the letter Z , i.e.

$$\omega_{XY \rightarrow Z}(w) = \{w_1 Z w_2 : w = w_1 X Y w_2\}.$$

Similarly,

$$\omega_{Z \rightarrow XY}(w) = \{w_1 X Y w_2 : w = w_1 Z w_2\}.$$

Clearly,

$$v \in \omega_{XY \rightarrow Z}(w) \Leftrightarrow w \in \omega_{Z \rightarrow XY}(v). \quad (9)$$

By abuse of notation, we write $v = \omega_{XY \rightarrow Z}(w)$ instead of $v \in \omega_{XY \rightarrow Z}(w)$.

Lemma 5.2. *Assume that the orbits of points α, β, γ and δ are mutually disjoint. For sufficiently small $\varepsilon > 0$, we have the following relations between I -itineraries of points in I*

- (a) $R(\mathfrak{A} - \varepsilon) = \omega_{B \rightarrow AC}(R(\mathfrak{A} + \varepsilon))$,
- (b) $R(\mathfrak{A} + \varepsilon) = \omega_{AC \rightarrow B}(R(\mathfrak{A} - \varepsilon))$,
- (c) $R(\mathfrak{B} - \varepsilon) = \omega_{CA \rightarrow B}(R(\mathfrak{B} + \varepsilon))$,
- (d) $R(\mathfrak{B} + \varepsilon) = \omega_{B \rightarrow CA}(R(\mathfrak{B} - \varepsilon))$,
- (e) $R(\mathfrak{D} + \varepsilon) = R(\mathfrak{D} - \varepsilon)R(\delta - \varepsilon)$,
- (f) $R(\mathfrak{C} - \varepsilon) = R(\mathfrak{C} + \varepsilon)R(\gamma + \varepsilon)$,

where $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ are given in Lemma 4.1.

Proof. We will first demonstrate the proof of the case a. Let $K = [\mathfrak{A} - \varepsilon, \mathfrak{A} + \varepsilon]$ with ε chosen such that $K \subset I$ and

$$\alpha, \beta, \gamma, \delta \notin T^i(K) \text{ for all } 0 \leq i \leq k_\alpha \text{ with the only exception of } T^{k_\alpha}(\mathfrak{A}) = \alpha. \quad (10)$$

For simplicity, denote $t = \max\{r_I(x) : x \in K\}$ the maximal return time. The existence of such ε follows trivially from the definition of the interval exchange transformation and the assumptions of the lemma.

Let $K_- = [\mathfrak{A} - \varepsilon, \mathfrak{A})$ and $K_+ = [\mathfrak{A}, \mathfrak{A} + \varepsilon]$. It follows from the definition of \mathfrak{A} and condition (10) that for all i such that $0 < i \leq k_\alpha$ we have $T^i(K) \cap I = \emptyset$. Moreover, condition (10) implies that all such $T^i(K)$ are intervals. It implies that for any $x, y \in K$, the prefixes of $R(x)$ and $R(y)$ of length $k_\alpha + 1$ are the same. Denote this prefix by w .

The definition of k_α implies that $\alpha \in T^{k_\alpha}(K)$. Since $T^{k_\alpha}(K_+) = [\alpha, \alpha + \varepsilon] \subset J_B$, we obtain

$$T^{k_\alpha+1}(K_+) = [T(\alpha), T(\alpha) + \varepsilon].$$

Furthermore, since $T^{k_\alpha}(K_-) = [\alpha - \varepsilon, \alpha] \subset J_A$, we obtain

$$T^{k_\alpha+1}(K_-) = [1 - \varepsilon, 1] \subset J_C,$$

and thus

$$T^{k_\alpha+2}(K_-) = [T(\alpha) - \varepsilon, T(\alpha)].$$

This implies that the set $K' = T^{k_\alpha+2}(K_-) \cup T^{k_\alpha+1}(K_+) = [T(\alpha) - \varepsilon, T(\alpha) + \varepsilon]$ is an interval. As above, condition (10) implies that the set $T^i(K')$ is an interval for all i such that $0 \leq i \leq t - k_\alpha - 1$. It follows that $\min\{i : T^i(K') \cap K \neq \emptyset\} = t - k_\alpha - 2$ and condition (10) moreover implies that $T^{t-k_\alpha-2}(K') \subset K$. Thus, the iterations $x, T(x), \dots, T^{t-k_\alpha-2}(x)$ of every $x \in K'$ are coded by the same word, say v .

The whole situation is depicted in Figure 1. From what is said above, we can write down the I -itineraries of points from K ,

$$R(x) = \begin{cases} wACv & \text{if } x \in K_-, \\ wBv & \text{if } x \in K_+. \end{cases}$$

This finishes the proof of (a).

The claim in item (c) is analogous to (a). Cases (b) and (d) are derived from (a) and (c) by the use of equivalence (9).

Let us now demonstrate the proof of the case (e). Denote $s = \min\{n \in \mathbb{Z}^+ : T^n(\delta) \in I\}$. Let $K = [\mathfrak{D} - \varepsilon, \mathfrak{D} + \varepsilon]$ with ε chosen such that $K \subset I$ and

$$\alpha, \beta, \gamma, \delta \notin T^i(K) \text{ for all } 0 \leq i \leq k_\delta + s \text{ with the only exception of } T^{k_\delta}(\mathfrak{D}) = \delta. \quad (11)$$

The existence of such ε follows trivially from the definition of the interval exchange transformation and the assumptions of the lemma.

Condition (11) implies that $T^i(K)$ is an interval for all i such that $0 < i \leq k_\delta + s$. Moreover, $T^i(K) \cap I = \emptyset$ for all i such that $0 < i < k_\delta$. We obtain $T^{k_\delta}(K) \cap I = [\delta - \varepsilon, \delta]$. In other words, the I -itineraries of all points of K start with a prefix of length k_δ which is equal to $R(\mathfrak{D} - \varepsilon)$. Condition (11) and the definition of s implies that for all i such that $k_\delta < i < s + k_\delta$ we have $T^i(K) \subset J_X$ for some $X \in \{A, B, C\}$ and $T^i(K) \cap I = \emptyset$. Moreover, $T^i(K) \subset I$ for $i = k_\delta + s$.

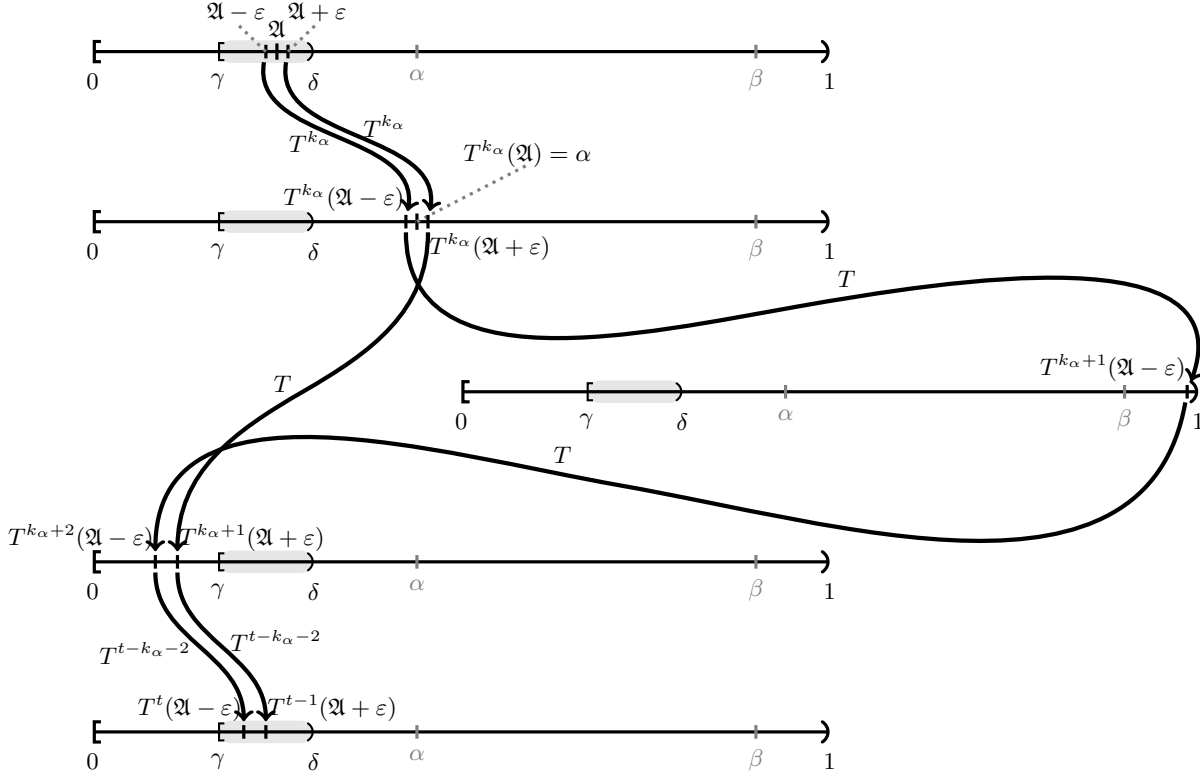


Figure 1: Situation in the proof of Lemma 5.2, case a.

Thus, the iterations of points of $T^{k_\delta}(K) = [\delta - \varepsilon, \delta) \cup T^{k_\delta}[\mathfrak{D}, \mathfrak{D} + \varepsilon]$ are coded by the same word of length s , namely $R(\delta - \varepsilon)$. Altogether, we can conclude that the I -itinerary of points in the interval $[\mathfrak{D}, \mathfrak{D} + \varepsilon]$ is equal to $R(\mathfrak{D} - \varepsilon)R(\delta - \varepsilon)$. The situation is depicted in Figure 2.

Case (f) can be treated in a way analogous to case (e). \square

Now we are in the state to prove the main theorem describing the return times in $\mathfrak{I}_{\text{iet}}$. In the proof, it is sufficient to focus on the case when $\#\text{It}_I = 5$, since, as we have seen from Proposition 3.3, the set of I -itineraries, and thus also their return times, for the other cases is only a subset of $\text{It}_{\tilde{I}}$ for some “close enough” generic subinterval $\tilde{I} \subset [0, 1)$. So throughout the rest of this section, suppose that $\#\text{It}_I = 5$. This means by Lemma 4.1 that points $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ lie in the interior of the interval $I = [\gamma, \delta)$ and are mutually distinct, moreover, by Proposition 4.2, we have $\mathfrak{D} < \mathfrak{C}$. Such conditions imply 12 possible orderings of $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ which give rise to 12 cases in the study of return times. We will describe them in the proof of Theorem 5.1 as cases (i)–(xii) and then show in Example 5.5 that all 12 cases may occur.

Remark 5.3. Note that if $\gamma = 0$, i.e. we induce on an interval $I = [0, \delta)$, we have $T^{-1}(\gamma) = \beta$ and therefore necessarily $\mathfrak{B} = \mathfrak{C}$. Thus there are at most four I -itineraries. Due to Proposition 3.4, similar situation happens if $\delta = 1$.

Proof of Theorem 5.1. We will discuss the 12 possibilities of ordering of points $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ in the interior of the interval $[\gamma, \delta)$ with the condition $\mathfrak{D} < \mathfrak{C}$. The structure of the set of I -itineraries will be best shown in terms of I -itineraries of points in the left neighbourhood of the point \mathfrak{D} and

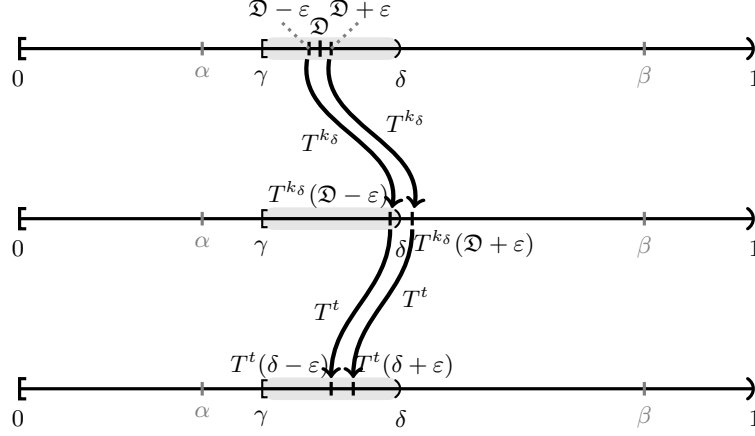


Figure 2: Situation in the proof of Lemma 5.2, case d.

right neighbourhood of the point \mathfrak{C} . For simplicity, we thus denote for sufficiently small positive ε

$$R_1 = R(\mathfrak{D} - \varepsilon), \quad R_2 = R(\mathfrak{C} + \varepsilon) \quad \text{and} \quad |R_1| = t_1, |R_2| = t_2.$$

In order to be allowed to use Lemma 5.2, we will assume that the orbits of points α, β, γ and δ are mutually disjoint. Otherwise, we use Proposition 3.3 to find a modified interval \tilde{I} where this is satisfied and $\text{It}_{\tilde{I}} = \text{It}_I$.

- (i) Let $\mathfrak{A} < \mathfrak{B} < \mathfrak{D} < \mathfrak{C}$. We know that $R(x)$ is constant on the intervals $[\gamma, \mathfrak{A})$, $[\mathfrak{A}, \mathfrak{B})$, $[\mathfrak{B}, \mathfrak{D})$, $[\mathfrak{D}, \mathfrak{C})$, and $[\mathfrak{C}, \delta)$. By definition $R(x) = R_2$ for $x \in [\mathfrak{C}, \delta)$ and $R(x) = R_1$ for $x \in [\mathfrak{B}, \mathfrak{D})$. We can derive from rule (e) of Lemma 5.2 that if $x \in [\mathfrak{D}, \mathfrak{C})$, then $R(x) = R_1 R_2$. Further, we use rule (c) to show that $R(x) = \omega_{CA \rightarrow B}(R_1)$ for $x \in [\mathfrak{A}, \mathfrak{B})$ and further by applying rule (a), we obtain that $R(x) = \omega_{B \rightarrow AC}(\omega_{CA \rightarrow B}(R_1))$ for $x \in [\gamma, \mathfrak{A})$. Summarized,

$$R(x) = \begin{cases} \omega_{B \rightarrow AC}(\omega_{CA \rightarrow B}(R_1)) & \text{for } x \in [\gamma, \mathfrak{A}) \\ \omega_{CA \rightarrow B}(R_1) & \text{for } x \in [\mathfrak{A}, \mathfrak{B}) \\ R_1 & \text{for } x \in [\mathfrak{B}, \mathfrak{D}) \\ R_1 R_2 & \text{for } x \in [\mathfrak{D}, \mathfrak{C}) \\ R_2 & \text{for } x \in [\mathfrak{C}, \delta). \end{cases}$$

It is easy to show that the lengths of the above I -itineraries are t_1 , $t_1 - 1$, t_1 , $t_1 + t_2$, t_2 , respectively. Setting $r_1 = t_1 - 1$ and $r_2 = t_2$, we obtain the desired return times.

The proofs of the other cases are analogous, we state the results in terms of R_1 and R_2 .

- (ii) Let $\mathfrak{D} < \mathfrak{C} < \mathfrak{A} < \mathfrak{B}$. We obtain

$$R(x) = \begin{cases} R_1 & \text{for } x \in [\gamma, \mathfrak{D}) \\ R_2 R_1 & \text{for } x \in [\mathfrak{D}, \mathfrak{C}) \\ R_2 & \text{for } x \in [\mathfrak{C}, \mathfrak{A}) \\ \omega_{AC \rightarrow B}(R_2) & \text{for } x \in [\mathfrak{A}, \mathfrak{B}) \\ \omega_{B \rightarrow CA}(\omega_{AC \rightarrow B}(R_2)) & \text{for } x \in [\mathfrak{B}, \delta), \end{cases}$$

with lengths t_1 , $t_1 + t_2$, t_2 , $t_2 - 1$, t_2 , respectively. We set $r_1 = t_1$ and $r_2 = t_2 - 1$.

(iii) Let $\mathfrak{B} < \mathfrak{A} < \mathfrak{D} < \mathfrak{C}$. A discussion as above leads to

$$R(x) = \begin{cases} \omega_{CA \rightarrow B}(\omega_{B \rightarrow AC}(R_1)) & \text{for } x \in [\gamma, \mathfrak{B}) \\ \omega_{B \rightarrow AC}(R_1) & \text{for } x \in [\mathfrak{B}, \mathfrak{A}) \\ R_1 & \text{for } x \in [\mathfrak{A}, \mathfrak{D}) \\ R_1 R_2 & \text{for } x \in [\mathfrak{D}, \mathfrak{C}) \\ R_2 & \text{for } x \in [\mathfrak{C}, \delta), \end{cases}$$

with the corresponding lengths $t_1, t_1 + 1, t_1, t_1 + t_2, t_2$, respectively. We set $r_1 = t_1$ and $r_2 = t_2$.

(iv) Let $\mathfrak{D} < \mathfrak{C} < \mathfrak{B} < \mathfrak{A}$. We obtain

$$R(x) = \begin{cases} R_1 & \text{for } x \in [\gamma, \mathfrak{D}) \\ R_2 R_1 & \text{for } x \in [\mathfrak{D}, \mathfrak{C}) \\ R_2 & \text{for } x \in [\mathfrak{C}, \mathfrak{B}) \\ \omega_{B \rightarrow CA}(R_2) & \text{for } x \in [\mathfrak{B}, \mathfrak{A}) \\ \omega_{AC \rightarrow B}(\omega_{B \rightarrow CA}(R_2)) & \text{for } x \in [\mathfrak{A}, \delta), \end{cases}$$

with lengths $t_1, t_1 + t_2, t_2, t_2 + 1, t_2$, respectively. We set $r_1 = t_1$ and $r_2 = t_2$.

(v) Let $\mathfrak{A} < \mathfrak{D} < \mathfrak{B} < \mathfrak{C}$. We obtain

$$R(x) = \begin{cases} \omega_{B \rightarrow AC}(R_1) & \text{for } x \in [\gamma, \mathfrak{A}) \\ R_1 & \text{for } x \in [\mathfrak{A}, \mathfrak{D}) \\ R_1 R_2 & \text{for } x \in [\mathfrak{D}, \mathfrak{B}) \\ R_2 \omega_{B \rightarrow AC}(R_1) & \text{for } x \in [\mathfrak{B}, \mathfrak{C}) \\ R_2 & \text{for } x \in [\mathfrak{C}, \delta), \end{cases}$$

with lengths $t_1 + 1, t_1, t_1 + t_2, t_1 + t_2 + 1, t_2$, respectively. We set $r_1 = t_1$ and $r_2 = t_2$.

(vi) Let $\mathfrak{D} < \mathfrak{A} < \mathfrak{C} < \mathfrak{B}$. We obtain

$$R(x) = \begin{cases} R_1 & \text{for } x \in [\gamma, \mathfrak{D}) \\ R_1 \omega_{B \rightarrow CA}(R_2) & \text{for } x \in [\mathfrak{D}, \mathfrak{A}) \\ R_2 R_1 & \text{for } x \in [\mathfrak{A}, \mathfrak{C}) \\ R_2 & \text{for } x \in [\mathfrak{C}, \mathfrak{B}) \\ \omega_{B \rightarrow CA}(R_2) & \text{for } x \in [\mathfrak{B}, \delta), \end{cases}$$

with lengths $t_1, t_1 + t_2 + 1, t_1 + t_2, t_2, t_2 + 1$, respectively. We set $r_1 = t_1$ and $r_2 = t_2$.

(vii) Let $\mathfrak{B} < \mathfrak{D} < \mathfrak{A} < \mathfrak{C}$. We obtain

$$R(x) = \begin{cases} \omega_{CA \rightarrow B}(R_1) & \text{for } x \in [\gamma, \mathfrak{B}) \\ R_1 & \text{for } x \in [\mathfrak{B}, \mathfrak{D}) \\ R_1 R_2 & \text{for } x \in [\mathfrak{D}, \mathfrak{A}) \\ R_2 \omega_{CA \rightarrow B}(R_1) & \text{for } x \in [\mathfrak{A}, \mathfrak{C}) \\ R_2 & \text{for } x \in [\mathfrak{C}, \delta), \end{cases}$$

with lengths $t_1 - 1, t_1, t_1 + t_2, t_1 + t_2 - 1, t_2$, respectively. We set $r_1 = t_1 - 1$ and $r_2 = t_2$.

(viii) Let $\mathfrak{D} < \mathfrak{B} < \mathfrak{C} < \mathfrak{A}$. We obtain

$$R(x) = \begin{cases} R_1 & \text{for } x \in [\gamma, \mathfrak{D}) \\ R_1 \omega_{AC \rightarrow B}(R_2) & \text{for } x \in [\mathfrak{D}, \mathfrak{B}) \\ R_2 R_1 & \text{for } x \in [\mathfrak{B}, \mathfrak{C}) \\ R_2 & \text{for } x \in [\mathfrak{C}, \mathfrak{A}) \\ \omega_{AC \rightarrow B}(R_2) & \text{for } x \in [\mathfrak{A}, \delta), \end{cases}$$

with lengths $t_1, t_1 + t_2 - 1, t_1 + t_2, t_2, t_2 - 1$, respectively. We set $r_1 = t_1$ and $r_2 = t_2 - 1$.

(ix) Let $\mathfrak{A} < \mathfrak{D} < \mathfrak{C} < \mathfrak{B}$. We obtain

$$R(x) = \begin{cases} \omega_{B \rightarrow AC}(R_1) & \text{for } x \in [\gamma, \mathfrak{A}) \\ R_1 & \text{for } x \in [\mathfrak{A}, \mathfrak{D}) \\ R_1 \omega_{B \rightarrow CA}(R_2) & \text{for } x \in [\mathfrak{D}, \mathfrak{C}) \\ R_2 & \text{for } x \in [\mathfrak{C}, \mathfrak{B}) \\ \omega_{B \rightarrow CA}(R_2) & \text{for } x \in [\mathfrak{B}, \delta), \end{cases}$$

with lengths $t_1 + 1, t_1, t_1 + t_2 + 1, t_2, t_2 + 1$, respectively. We set $r_1 = t_1$ and $r_2 = t_2$.

(x) Let $\mathfrak{B} < \mathfrak{D} < \mathfrak{C} < \mathfrak{A}$. We obtain

$$R(x) = \begin{cases} \omega_{CA \rightarrow B}(R_1) & \text{for } x \in [\gamma, \mathfrak{B}) \\ R_1 & \text{for } x \in [\mathfrak{B}, \mathfrak{D}) \\ R_1 \omega_{AC \rightarrow B}(R_2) & \text{for } x \in [\mathfrak{D}, \mathfrak{C}) \\ R_2 & \text{for } x \in [\mathfrak{C}, \mathfrak{A}) \\ \omega_{AC \rightarrow B}(R_2) & \text{for } x \in [\mathfrak{A}, \delta), \end{cases}$$

with lengths $t_1 - 1, t_1, t_1 + t_2 - 1, t_2, t_2 - 1$, respectively. We set $r_1 = t_1 - 1$ and $r_2 = t_2 - 1$.

(xi) Let $\mathfrak{D} < \mathfrak{A} < \mathfrak{B} < \mathfrak{C}$. We obtain

$$R(x) = \begin{cases} R_1 & \text{for } x \in [\gamma, \mathfrak{D}) \\ R_1 R_2 & \text{for } x \in [\mathfrak{D}, \mathfrak{A}) \\ \omega_{CA \rightarrow B}(R_2 R_1) & \text{for } x \in [\mathfrak{A}, \mathfrak{B}) \\ R_2 R_1 & \text{for } x \in [\mathfrak{B}, \mathfrak{C}) \\ R_2 & \text{for } x \in [\mathfrak{C}, \delta), \end{cases}$$

with lengths $t_1, t_1 + t_2, t_1 + t_2 - 1, t_1 + t_2, t_2$, respectively. We set $r_1 = t_1 - 1$ and $r_2 = t_2$.

(xii) Let $\mathfrak{D} < \mathfrak{B} < \mathfrak{A} < \mathfrak{C}$. We obtain

$$R(x) = \begin{cases} R_1 & \text{for } x \in [\gamma, \mathfrak{D}) \\ R_1 R_2 & \text{for } x \in [\mathfrak{D}, \mathfrak{B}) \\ \omega_{B \rightarrow AC}(R_2 R_1) & \text{for } x \in [\mathfrak{B}, \mathfrak{A}) \\ R_2 R_1 & \text{for } x \in [\mathfrak{A}, \mathfrak{C}) \\ R_2 & \text{for } x \in [\mathfrak{C}, \delta), \end{cases}$$

with lengths $t_1, t_1 + t_2, t_1 + t_2 + 1, t_1 + t_2, t_2$, respectively. We set $r_1 = t_1$ and $r_2 = t_2$.

γ	δ	type	lengths
$\frac{6}{25}$	$\frac{99}{100}$	$\mathfrak{A} < \mathfrak{B} < \mathfrak{D} < \mathfrak{C}$	[2, 1, 2, 3, 1]
$\frac{29}{100}$	$\frac{71}{100}$	$\mathfrak{D} < \mathfrak{C} < \mathfrak{A} < \mathfrak{B}$	[1, 15, 14, 13, 14]
$\frac{77}{100}$	$\frac{4}{5}$	$\mathfrak{B} < \mathfrak{A} < \mathfrak{D} < \mathfrak{C}$	[88, 89, 88, 109, 21]
$\frac{7}{25}$	$\frac{3}{4}$	$\mathfrak{D} < \mathfrak{C} < \mathfrak{B} < \mathfrak{A}$	[1, 13, 12, 13, 12]
$\frac{1}{100}$	$\frac{3}{4}$	$\mathfrak{A} < \mathfrak{D} < \mathfrak{B} < \mathfrak{C}$	[2, 1, 2, 3, 1]
$\frac{1}{100}$	$\frac{29}{100}$	$\mathfrak{D} < \mathfrak{A} < \mathfrak{C} < \mathfrak{B}$	[2, 14, 13, 11, 12]
$\frac{1}{4}$	$\frac{99}{100}$	$\mathfrak{B} < \mathfrak{D} < \mathfrak{A} < \mathfrak{C}$	[1, 2, 3, 2, 1]
$\frac{71}{100}$	$\frac{99}{100}$	$\mathfrak{D} < \mathfrak{B} < \mathfrak{C} < \mathfrak{A}$	[2, 13, 14, 12, 11]
$\frac{1}{25}$	$\frac{37}{50}$	$\mathfrak{A} < \mathfrak{D} < \mathfrak{C} < \mathfrak{B}$	[2, 1, 4, 2, 3]
$\frac{29}{100}$	$\frac{99}{100}$	$\mathfrak{B} < \mathfrak{D} < \mathfrak{C} < \mathfrak{A}$	[1, 2, 4, 3, 2]
$\frac{1}{100}$	$\frac{99}{100}$	$\mathfrak{D} < \mathfrak{A} < \mathfrak{B} < \mathfrak{C}$	[1, 2, 1, 2, 1]
$\frac{1}{4}$	$\frac{3}{4}$	$\mathfrak{D} < \mathfrak{B} < \mathfrak{A} < \mathfrak{C}$	[1, 12, 13, 12, 11]

Table 1: The cases (i)–(xii) from the proof of Theorem 5.1 for $\alpha = \frac{1}{5}\sqrt{5} - \frac{1}{5}$, $\beta = -\frac{1}{6}\sqrt{5} + \frac{2}{3}$ as in Example 5.5. The endpoints of the interval $I = [\gamma, \delta)$ are in the first and second column. The last column contains a list of lengths of I -itineraries of all 5 subintervals of I starting from the leftmost one.

□

Remark 5.4. When describing the I -itineraries using the words R_1, R_2 , we could apply the rules of Lemma 5.2 in a different order. By doing so, we would obtain the itineraries expressed differently, which yields interesting relations between words R_1, R_2 . For example, in the case (ix), we derive that the I -itinerary of $x \in [\mathfrak{D}, \mathfrak{C})$ is $R(x) = R_1\omega_{B \rightarrow CA}(R_2) = R_2\omega_{B \rightarrow AC}(R_1)$.

Note also the symmetries between the cases (i) and (ii), (iii) and (iv), (v) and (vi), (vii) and (viii), in consequence of Proposition 3.4. Indeed, if we exchange pair of points $\mathfrak{D} \leftrightarrow \mathfrak{C}$, $\mathfrak{B} \leftrightarrow \mathfrak{A}$, letters $A \leftrightarrow C$, and finally the inequalities $<$ and $>$, we obtain a symmetric situation in the list of cases we discussed in the proof. In this sense, each of cases (ix) up to (xii) is symmetric to itself.

Example 5.5. Set $\alpha = \frac{1}{5}\sqrt{5} - \frac{1}{5}$ and $\beta = -\frac{1}{6}\sqrt{5} + \frac{2}{3}$. Table 1 shows 12 choices of $I = [\gamma, \delta)$ which produce 12 distinct orders of the points $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ and \mathfrak{D} , shown in the third column. The last column contains the respective lengths of the 5 distinct I -itineraries.

Let us describe in detail one of the cases, namely the case $\mathfrak{B} < \mathfrak{D} < \mathfrak{C} < \mathfrak{A}$. The induced

interval is determined by setting $\gamma = \frac{29}{100}$ and $\delta = \frac{99}{100}$. One can verify that

$$\begin{aligned}\mathfrak{B} &= T^{-0}(\beta) = \frac{1}{6}\sqrt{5} + \frac{1}{3} \approx 0.706011329583298; \\ \mathfrak{D} &= T^{-2}(\delta) = \frac{11}{30}\sqrt{5} + \frac{37}{300} \approx 0.943224925083256; \\ \mathfrak{C} &= T^{-3}(\gamma) = \frac{8}{15}\sqrt{5} - \frac{73}{300} \approx 0.949236254666554; \\ \mathfrak{A} &= T^{-1}(\alpha) = \frac{11}{30}\sqrt{5} + \frac{2}{15} \approx 0.953224925083256.\end{aligned}$$

It corresponds to the case (x) in the proof of Theorem 5.1 with $R_1 = CA$ and $R_2 = CAC$. The I -itinerary of a point $x \in I = [\gamma, \delta)$ is

$$R(x) = \begin{cases} B & \text{for } x \in [\gamma, \mathfrak{B}) \\ CA & \text{for } x \in [\mathfrak{B}, \mathfrak{D}) \\ CACB & \text{for } x \in [\mathfrak{D}, \mathfrak{C}) \\ CAC & \text{for } x \in [\mathfrak{C}, \mathfrak{A}) \\ CB & \text{for } x \in [\mathfrak{A}, \delta) \end{cases}$$

6 Gaps and distance theorems

Let us reinterpret the statement of Theorem 5.1 in point of view of three gap and three distance theorems which are narrowly connected with exchange of two intervals. Under the name three gap theorem one usually refers to the description of gaps between neighbouring elements of the set

$$\mathcal{G}(\alpha, \delta) := \{n \in \mathbb{N} : \{n\alpha\} < \delta\} \subset \mathbb{N},$$

where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\delta \in (0, 1)$ and $\{x\} = x - \lfloor x \rfloor$ stands for the fractional part of x , see [28]. Sometimes one uses a more general formulation, namely the set

$$\mathcal{G}(\alpha, \rho, \gamma, \delta) := \{n \in \mathbb{N} : \gamma \leq \{n\alpha + \rho\} < \delta\} \subset \mathbb{N},$$

where moreover $\rho \in \mathbb{N}$, $0 \leq \gamma < \delta < 1$. The three gap theorem states that there exist integers r_1, r_2 such that gaps between neighbours in $\mathcal{G}(\alpha, \rho, \gamma, \delta)$ take at most three values, namely in the set $\{r_1, r_2, r_1 + r_2\}$.

Let us interpret the three gap theorem in the frame of exchange of two intervals $J_0 = [0, 1 - \alpha)$, $J_1 = [1 - \alpha, 1)$. The transformation $T : [0, 1) \rightarrow [0, 1)$ is of the form

$$T(x) = \begin{cases} x + \alpha & \text{for } x \in [0, 1 - \alpha), \\ x + \alpha - 1 & \text{for } x \in [1 - \alpha, 1), \end{cases} \quad \text{i.e.} \quad T(x) = \{x + \alpha\}.$$

Therefore we can write

$$\mathcal{G}(\alpha, \rho, \gamma, \delta) := \{n \in \mathbb{N} : T^n(\rho) \in [\gamma, \delta)\}, \quad (12)$$

and the gaps in this set correspond to return times to the interval $[\gamma, \delta)$ under the transformation T .

Our Theorem 5.1 is an analogue of the three gap theorem in the form (12) generalized for the case when the transformation T is a non-degenerate 3iet. We see that there are 5 gaps, but still expressed using two basic values r_1, r_2 .

The so-called three distance theorem focuses on distances between neighbours of the set

$$\mathcal{D}(\alpha, \rho, N) := \{\{\alpha n + \rho\} : n \in \mathbb{N}, n < N\} \subset [0, 1).$$

The three distance theorem ensures existence of $\Delta_1, \Delta_2 > 0$ such that distances between neighbours in $\mathcal{D}(\alpha, \rho, N)$ take at most three values, namely in $\{\Delta_1, \Delta_2, \Delta_1 + \Delta_2\}$.

In the framework of 2iet T , we can write for the distances

$$\mathcal{D}(\alpha, \rho, N) := \{T^n(\rho) : n \in \mathbb{N}, n < N\} \subset [0, 1). \quad (13)$$

We could try to study the analogue of the three distance theorem in the form (13) for exchanges of three intervals. In fact, it can be derived from the results of [12] that if T is a 3iet with discontinuity points α, β , then

$$\mathcal{D}(\alpha, \beta, \rho, N) := \{T^n(\rho) : n \in \mathbb{N}, n < N\}$$

has again at most three distances Δ_1, Δ_2 , and $\Delta_1 + \Delta_2$ for some positive Δ_1, Δ_2 .

The three distance theorem can also be used to derive that the frequencies of factors of length n in a Sturmian word take at most three values. Recall that the frequency of a factor w in the infinite word $\mathbf{u} = u_0 u_1 u_2 \dots$ is given by

$$\text{freq}(w) := \lim_{N \rightarrow \infty} \frac{1}{N} (\#\{0 \leq i < N : w \text{ is a prefix of } u_i u_{i+1} \dots\}),$$

if the limit exists.

It is a well known fact that the frequencies of factors of length n in a coding of an exchange of intervals are given by the lengths of cylinders corresponding to the factors. The boundary points of these cylinders are $T^{-j}(1 - \alpha)$, for $j = 0, \dots, n - 1$. Consequently, the distances in the set $\mathcal{D}(\alpha, 1 - \alpha, N)$ are precisely the frequencies of factors, and the three distance theorem implies the well known fact that Sturmian words have for each n only three values of frequencies of factors of length n , namely $\varrho_1, \varrho_2, \varrho_1 + \varrho_2$.

The frequencies of factors of length n in 3iet words are given by distances between neighbours of the set

$$\{T^{-n}(\alpha) : n \in \mathbb{N}, n < N\} \cup \{T^{-n}(\beta) : n \in \mathbb{N}, n < N\}.$$

In [7] it is shown, based on the study of Rauzy graphs, that the number of distinct values of frequencies in infinite words with reversal closed language satisfies

$$\#\{\text{freq}(w) : w \in \mathcal{L}(\mathbf{u}), |w| = n\} \leq 2(\mathcal{C}_{\mathbf{u}}(n) - \mathcal{C}_{\mathbf{u}}(n - 1)) + 1,$$

which in case of 3iet words reduces to ≤ 5 . Paper [11] shows, that the set of integers n for which this bound is achieved, is of density 1 in \mathbb{N} .

7 Description of the case of three I -itineraries

The cases (i) – (xii) in the proof of Theorem 5.1 correspond to the generic instances of a subinterval I in a non-degenerate 3iet which lead to 5 different I -itineraries. Let us focus on the cases where, on the contrary, the set of I -itineraries has only 3 elements. First we recall two reasons why such cases are interesting.

For a factor w from the language of a non-degenerate 3iet transformation T , denote

$$[w] = \{\rho \in [0, 1) : w \text{ is a prefix of } \mathbf{u}_\rho\}.$$

It is easy to see that $[w]$ – usually called the cylinder of w – is a semi-closed interval and its boundaries belong to the set $\{T^{-i}(z) : 0 \leq i < n, z \in \{\alpha, \beta\}\}$. Clearly, a factor v is a return word to the factor w if and only if v is a $[w]$ -itinerary. It is well known [33] that any factor of an infinite word coding a non-degenerate 3iet has exactly three return words and thus the set $It_{[w]}$ has three elements.

The second reason why to study intervals I yielding three I -itineraries is that any morphism fixing a non-degenerate 3iet word corresponds to such an interval I . Details of this correspondence will be explained in Section 8.

Proposition 7.1. *Let T be a non-degenerate 3iet and let $I = [\gamma, \delta) \subset [0, 1)$ be such that $\#It_I = 3$. One of the following cases occurs:*

(i) $\mathfrak{B} = \mathfrak{D} < \mathfrak{A} = \mathfrak{C}$ and

$$R(x) = \begin{cases} R_1 & \text{for } x \in [\gamma, \mathfrak{B}) \\ \omega_{B \rightarrow CA}(R_1 R_2) = \omega_{B \rightarrow AC}(R_2 R_1) & \text{for } x \in [\mathfrak{B}, \mathfrak{A}) \\ R_2 & \text{for } x \in [\mathfrak{A}, \delta) \end{cases}$$

(ii) $\mathfrak{A} = \mathfrak{D} < \mathfrak{B} = \mathfrak{C}$ and

$$R(x) = \begin{cases} R_1 & \text{for } x \in [\gamma, \mathfrak{A}) \\ \omega_{AC \rightarrow B}(R_1 R_2) = \omega_{CA \rightarrow B}(R_2 R_1) & \text{for } x \in [\mathfrak{A}, \mathfrak{B}) \\ R_2 & \text{for } x \in [\mathfrak{B}, \delta) \end{cases}$$

(iii) $\mathfrak{B} < \mathfrak{A} = \mathfrak{C} = \mathfrak{D}$ and

$$R(x) = \begin{cases} \omega_{CA \rightarrow B}(R_1) & \text{for } x \in [\gamma, \mathfrak{B}) \\ R_1 & \text{for } x \in [\mathfrak{B}, \mathfrak{A}) \\ R_2 & \text{for } x \in [\mathfrak{A}, \delta) \end{cases}$$

(iv) $\mathfrak{B} = \mathfrak{C} = \mathfrak{D} < \mathfrak{A}$ and

$$R(x) = \begin{cases} R_1 & \text{for } x \in [\gamma, \mathfrak{A}) \\ R_2 & \text{for } x \in [\mathfrak{A}, \mathfrak{B}) \\ \omega_{AC \rightarrow B}(R_2) & \text{for } x \in [\mathfrak{B}, \delta) \end{cases}$$

(v) $\mathfrak{A} = \mathfrak{C} = \mathfrak{D} < \mathfrak{B}$ and

$$R(x) = \begin{cases} R_1 & \text{for } x \in [\gamma, \mathfrak{A}) \\ R_2 & \text{for } x \in [\mathfrak{A}, \mathfrak{B}) \\ \omega_{B \rightarrow CA}(R_2) & \text{for } x \in [\mathfrak{B}, \delta) \end{cases}$$

(vi) $\mathfrak{A} < \mathfrak{B} = \mathfrak{C} = \mathfrak{D}$ and

$$R(x) = \begin{cases} \omega_{B \rightarrow AC}(R_1) & \text{for } x \in [\gamma, \mathfrak{A}) \\ R_1 & \text{for } x \in [\mathfrak{A}, \mathfrak{B}) \\ R_2 & \text{for } x \in [\mathfrak{B}, \delta) \end{cases}$$

Sketch of a proof. Since by Lemma 4.1 the subintervals of I corresponding to the same itinerary are delimited by the points $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ and \mathfrak{D} , we may have $\#It_I = 3$ only if some of these points coincide, more precisely if $\#\{\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}\} = 2$. The non-degeneracy of the considered 3iet implies that always $\mathfrak{A} \neq \mathfrak{B}$, which further limits the discussion.

The six cases listed in the statement are the possibilities of how this may happen, respecting the condition $\mathfrak{D} < \mathfrak{C}$ or $\mathfrak{D} = \mathfrak{C}$. In order to describe the itineraries, denote again

$$R_1 = R(\mathfrak{D} - \varepsilon) \quad \text{and} \quad R_2 = R(\mathfrak{C} + \varepsilon)$$

for $\varepsilon > 0$ sufficiently small. One can then follow the ideas of the proof of Lemma 5.2. \square

Let us apply Proposition 7.1 in order to provide the description of return words to factors of a non-degenerate 3iet word. If a factor w has a unique right prolongation in the language $\mathcal{L}(T)$, i.e. there exists only one letter $a \in \mathcal{A}$ such that $wa \in \mathcal{L}(T)$, then the set of return words to w and the set of return words to wa coincide. And (almost) analogously, if a factor w has a unique left prolongation in the language $\mathcal{L}(T)$, say aw for some $a \in \mathcal{A}$, then a word v is a return word to w if and only if ava^{-1} is a return word to aw . Consequently, to describe the structure of return words to a given factor w , we can restrict to factors which have at least two right and at least two left prolongations. Such factors are called *bispecial*. It is readily seen that the language of an aperiodic recurrent infinite word \mathbf{u} contains infinitely many bispecial factors. Before giving the description of return words to bispecial factors, we state the following lemma.

Lemma 7.2. *Let w belong to language of a non-degenerate 3iet T . Denote $n = |w|$. For the cylinder of its reversal \overline{w} , one has*

$$[\overline{w}] = \overline{T^n([w])}.$$

Proof. According to definition of $[w]$, for each $[w]$ -itinerary r , the word rw belongs to the language and w occurs in rw exactly twice, as a prefix and as a suffix. In other words r is a return word to w . Moreover, $[w]$ is the maximal (with respect to inclusion) interval with this property. According to Remark 3.2, if r is an $[w]$ -itinerary, then the word $w^{-1}rw$ is an $T^n([w])$ -itinerary. Applying Proposition 3.4 to the interval $T^n([w])$ we obtain that $s := \overline{w^{-1}rw}$ is an $\overline{T^n([w])}$ -itinerary. Since the word $s\overline{w} = \overline{rw}$ has a prefix \overline{w} and a suffix \overline{w} , with no other occurrences of \overline{w} , the word s is a return word to \overline{w} and thus by definition of the cylinder, $s = \overline{w^{-1}rw}$ belongs to $[\overline{w}]$ -itinerary for any $\overline{T^n([w])}$ -itinerary s . From the maximality of the cylinder we have $\overline{T^n([w])} \subset [\overline{w}]$. Since lengths of the intervals $[w]$ and $T^n([w])$ coincide we have, in particular, that the length of interval $[w]$ is less or equal to the length of the interval $[\overline{w}]$. But from the symmetry of the role w and \overline{w} , their length must be equal and thus $\overline{T^n([w])} = [\overline{w}]$. \square

The language of T contains two types of bispecial factors: palindromic and non-palindromic. In [10], Ferenczi, Holton and Zamboni studied the structure of return words to non-palindromic bispecial factors. The following proposition completes this description.

Proposition 7.3. *Let w be a bispecial factor. If w is a palindrome, then its return words are described by the cases (i) and (ii) of Proposition 7.1. If w is not a palindrome, then its return words are described by the cases (iii) – (iv) of Proposition 7.1.*

Proof. Let w be a bispecial factor. If w is not a palindrome, the claim follows from Theorem 4.6 of [10].

Assume w is a palindrome and let $[w] = [T^{-\ell}(L), T^{-r}(R)]$ with $L, R \in \{\alpha, \beta\}$ and $0 \leq \ell, r < |w|$. By Lemma 7.2 we have $[\overline{w}] = \overline{T^{|w|}([w])}$. Since $w = \overline{w}$, we have $I_w = I_{\overline{w}}$, and thus

$$I_w = [T^{-\ell}(L), T^{-r}(R)] = [1 - T^{n-r}(R), 1 - T^{n-\ell}(L)] = I_{\overline{w}}.$$

Since the considered 3iet is non-degenerate, the parameters α, β satisfy (8). Consequently, the equation $T^{-\ell}(L) = 1 - T^{n-r}(R)$ has a solution if and only if $R \neq L$. Thus, we have neither $\mathfrak{A} = \mathfrak{C} = \mathfrak{D}$ nor $\mathfrak{B} = \mathfrak{C} = \mathfrak{D}$ and we are in the case (i) or (ii) of Proposition 7.1. \square

8 Substitution invariance and conjugation of substitutions

Let us recall the relation of induction to a subinterval I to substitution invariance of 3iet words. Let I be an interval $I \subset [0, 1)$ such that the set It_I of I -itineraries has three elements, say R_A , R_B and R_C . For every $\rho \in I$, the infinite word \mathbf{u}_ρ can be written as a concatenation of words R_A , R_B and R_C . For a letter $Y \in \{A, B, C\}$ denote $I_Y = \{x \in I : R(x) = R_Y\}$. Obviously, $I = I_A \cup I_B \cup I_C$, and the induced mapping T_I is an exchange of these three intervals. The order of the words R_A , R_B and R_C in the concatenation is determined by the iterations of $T_I(\rho)$.

Suppose that T_I is homothetic to T . Recall that mappings $f : I_f \rightarrow I_f$ and $g : I_g \rightarrow I_g$ are homothetic if there exists an affine bijection $\Phi : I_f \rightarrow I_g$ with $\Phi(x) = \lambda x + \mu$ such that

$$\Phi f(x) = g\Phi(x) \quad \text{for all } x \in I_f. \quad (14)$$

This means that f and g behave in the same way, up to a scaling factor λ and a shift μ of the domains I_f and I_g . In other words, the graphs of the mappings f and g are the same, up to their scale and placing. The homothety of T and T_I implies that $\Phi(J_Y) = I_Y$ for all $Y \in \{A, B, C\}$. From (14), we derive for every $k \in \mathbb{N}$ that $\Phi T^k(x) = T_I^k \Phi(x)$ for $x \in [0, 1)$, and thus $\Phi T^k(\rho) = T_I^k(\rho)$ whenever

$$\Phi(\rho) = \rho, \quad (15)$$

i.e., ρ is the homothety center. From the relation $\Phi(J_Y) = I_Y$ it follows that the k -th element in the concatenation of itineraries R_A , R_B and R_C is equal to R_Y if and only if the k -th letter in the infinite word \mathbf{u}_ρ is equal to Y . This is equivalent to saying that the infinite word \mathbf{u}_ρ is invariant under the substitution η given by

$$\eta(A) = R_A, \quad \eta(B) = R_B, \quad \eta(C) = R_C. \quad (16)$$

We conclude that the existence of an interval I with three itineraries and T_I homothetic to T leads to a substitution fixing a 3iet word whose intercept is the homothety center ρ . In fact, the converse holds, too, as shown in [5]. We summarize both statements as follows.

Theorem 8.1 ([5]). *Let ξ be a primitive substitution over $\{A, B, C\}$ with incidence matrix M and let T be a non-degenerate 3iet. The substitution ξ fixes the word \mathbf{u}_ρ coding the orbit of a point $\rho \in [0, 1)$ under T if and only if there exists an interval $I \subset [0, 1)$ with I -itineraries $\text{It}_I = \{R_A, R_B, R_C\}$ such that T_I is homothetic to T , ρ is the homothety center, and the substitution η given by*

$$\eta = \begin{cases} \xi & \text{if no eigenvalue of } M \text{ belongs to } (-1, 0), \\ \xi^2 & \text{otherwise,} \end{cases}$$

satisfies $\eta(A) = R_A$, $\eta(B) = R_B$ and $\eta(C) = R_C$.

Let us mention that the scaling factor $\lambda \in (0, 1)$ in the homothety mapping $\Phi(x) = \lambda x + \mu$ is equal to the length of the interval $I = [\gamma, \delta)$, i.e., $\lambda = \delta - \gamma$, and the shift μ is equal to the left end-point of the interval I , namely γ . Moreover, it is related to the intercept ρ of an infinite word \mathbf{u}_ρ in the following way: one has $\mu = \gamma = \rho(1 - \lambda)$, as follows from (15). In fact, λ is an

eigenvalue of the incidence matrix of η . It follows from [5] that if ξ has such an eigenvalue, then the choice $\eta = \xi$ is sufficient. Otherwise, the incidence matrix of ξ^2 has such an eigenvalue.

By Theorem 8.1, if \mathbf{u}_ρ is invariant under a substitution, we find an interval I such that T_I is homothetic to T . If $I' = T(I)$ is again an interval, then $T_{I'}$ is also homothetic to T , and the I' -itineraries change with respect to the I -itineraries, as described in Remark 3.2. To show the relation of the corresponding substitutions, we need the following definition.

Definition 8.2. Let φ and ψ be morphisms over \mathcal{A}^* and let $w \in \mathcal{A}^*$ be a word such that $w\varphi(a) = \psi(a)w$ for every letter $a \in \mathcal{A}$. The morphism φ is said to be a *left conjugate* of ψ and ψ is said to be a *right conjugate* of φ . We write $\varphi \triangleleft \psi$. If φ is a left or right conjugate of ψ , then we say φ is *conjugate* to ψ . If the only left conjugate of φ is φ itself, then φ is called the *leftmost conjugate* of ψ and we write $\varphi = \varphi_L$. If the only right conjugate of ψ is ψ itself, then ψ is called the *rightmost conjugate* of φ and we write $\psi = \varphi_R$.

Note that given a substitution ξ , its leftmost and rightmost conjugates ξ_L and ξ_R may not exist. If this happens, it can be shown that its fixed point is a periodic word. All the substitutions considered here thus possess their leftmost and rightmost conjugates.

Proposition 8.3. Let \mathbf{u}_ρ be a 3iet word coding the orbit of the point $\rho \in [0, 1)$ under a non-degenerate 3iet T . Moreover, assume that \mathbf{u}_ρ is a fixed point of a primitive substitution η such that the corresponding interval I of Theorem 8.1 is of length λ . Let η' be a left conjugate of η , i.e., $\eta(a)w = w\eta'(a)$ for some word $w \in \mathcal{A}^*$. The morphism η' fixes the infinite word $\mathbf{u}_{\rho'}$ with ρ' satisfying

$$(1 - \lambda)\rho' = T^n((1 - \lambda)\rho), \quad \text{where } n = |w|. \quad (17)$$

Moreover, the interval I' corresponding to η' by Theorem 8.1 satisfies $I' = T^n(I)$.

Proof. Suppose that w is a letter, i.e., $w = X \in \mathcal{A}$. Necessarily, the words $\eta(a)$ start with the letter X for all $a \in \mathcal{A}$. This means for the interval I that $I \subset J_X$. According to Remark 3.2, the interval $I' = T(I)$ has three I' -itineraries. Moreover, the induced mapping $T_{I'}$ is also homothetic to T . Denote $I = [\gamma, \delta)$. The homothety between the transformations T and T_I is achieved by the map $\Phi(x) = \lambda x + \gamma$. The homothety between T and $T_{I'}$ is the map $\Phi'(x) = \lambda x + T(\gamma)$. Since the intercepts ρ and ρ' are by (15) fixed by the homotheties Φ and Φ' , respectively, we have

$$\Phi(\rho) = \lambda\rho + \gamma = \rho \quad \text{and} \quad \Phi'(\rho') = \lambda\rho' + T(\gamma) = \rho'. \quad (18)$$

Eliminating γ , we obtain

$$(1 - \lambda)\rho' = T(\gamma) = T((1 - \lambda)\rho).$$

Since conjugation by any word w can be performed letter by letter, the proof is finished. \square

Remark 8.4. Note that the first of equalities in (18) implies for the left boundary point γ of the interval I that $\gamma = (1 - \lambda)\rho$. If w is as in Proposition 8.3, then for $0 \leq k < n = |w|$, the iterations $T^k(I)$ are all intervals, and hence the coding \mathbf{u}_x of every point $x \in I$ starts with the same prefix w .

In the following, we will also need to see the relation of the substitution η corresponding to the interval $I = [\gamma, \delta)$ with the substitution corresponding to the interval $\bar{I} = [1 - \delta, 1 - \gamma)$. It turns out that it is the mirror substitution of η , defined in general as follows. For a morphism $\xi : \mathcal{A} \rightarrow \mathcal{A}$, we define the morphism $\bar{\xi} : \mathcal{A} \rightarrow \mathcal{A}$ by $\bar{\xi}(a) = \xi(a)$ for $a \in \mathcal{A}$.

Proposition 8.5. *Let $I \subset [0, 1)$ be a left-closed right-open interval such that $\#It_I = 3$ and T_I is an exchange of three intervals with the permutation (321). The interval \bar{I} satisfies $\#It_{\bar{I}} = 3$ and the induced map $T_{\bar{I}}$ is homothetic to T_I . If, moreover, T_I is homothetic to T and the substitution η corresponding to I fixes the infinite word \mathbf{u}_ρ , then the substitution corresponding to \bar{I} is $\bar{\eta}$ and fixes the infinite word $\mathbf{u}_{\bar{\rho}}$, where $\bar{\rho} = 1 - \rho$.*

Proof. Denote $It_I = \{R_1, R_2, R_3\}$ and $I_j = \{x \in I : R_I(x) = R_j\} = [\gamma_j, \delta_j)$ for $j = 1, 2, 3$ so that $I_1 < I_2 < I_3$. By Proposition 3.4, the \bar{I} -itineraries are \bar{R}_1, \bar{R}_2 and \bar{R}_3 , where

$$I'_j = \{x \in \bar{I} : R_{\bar{I}}(x) = \bar{R}_j\} = [1 - \delta'_j, 1 - \gamma'_j),$$

where $[\gamma'_j, \delta'_j) = T_I[\gamma_j, \delta_j)$. Since T_I is an exchange of three intervals with the permutation (321) we have

$$T_I[\gamma_1, \delta_1) > T_I[\gamma_2, \delta_2) > T_I[\gamma_3, \delta_3),$$

and therefore $I'_1 < I'_2 < I'_3$. The induced map $T_{\bar{I}}$ is therefore an exchange of three intervals I'_1, I'_2 and I'_3 with the permutation (321) and since $|I'_j| = |I_j|$ for $j = 1, 2, 3$, the transformation $T_{\bar{I}}$ is homothetic to T_I .

Suppose that T_I is homothetic to the original 3iet T . By Theorem 8.1, there is a substitution η corresponding to the interval I and satisfying $\eta(A) = R_1$, $\eta(B) = R_2$ and $\eta(C) = R_3$. The mapping $T_{\bar{I}}$ is homothetic to T_I and thus also to T , the corresponding substitution η' satisfies $\eta'(A) = \bar{R}_1$, $\eta'(B) = \bar{R}_2$ and $\eta'(C) = \bar{R}_3$. We can see that $\eta' = \bar{\eta}$.

Let ρ be the intercept of the infinite word which is fixed by the substitution η . It is the center of homothety between T_I and T , i.e., it is the fixed point of the mapping $\Phi(x) = (\delta - \gamma)x + \gamma$. We have $\rho = (\delta - \gamma)\rho + \gamma$, which implies

$$\rho = \frac{\gamma}{1 - \delta + \gamma}.$$

Similarly, the intercept $\bar{\rho}$ of the substitution $\bar{\eta}$ satisfies $\bar{\rho} = (\delta - \gamma)\bar{\rho} + 1 - \delta$, whence

$$\bar{\rho} = \frac{1 - \delta}{1 - \delta + \gamma} = 1 - \rho. \quad \square$$

For a finite word w , $\text{Fst}(w)$ and $\text{Lst}(w)$ denote the first and last letters of w , respectively.

Remark 8.6. Let η be a primitive substitution given by Theorem 8.1 fixing a 3iet word. Necessarily, the first and the last letters of $\eta(A)$, $\eta(B)$ and $\eta(C)$ satisfy

$$\text{Fst}(\eta(A)) \leq \text{Fst}(\eta(B)) \leq \text{Fst}(\eta(C)) \text{ and } \text{Lst}(\eta(A)) \leq \text{Lst}(\eta(B)) \leq \text{Lst}(\eta(C)),$$

where we consider the order $A < B < C$. The inequalities for the first letters follow from the definition of an exchange of intervals, namely from the fact that the words $\eta(A)$, $\eta(B)$ and $\eta(C)$ are given as I -itineraries. By Proposition 8.5, the last letters of the words $\eta(A)$, $\eta(B)$ and $\eta(C)$ are the first letters of the words $\bar{\eta}(A)$, $\bar{\eta}(B)$ and $\bar{\eta}(C)$ which proves the second set of inequalities.

9 Ternarization

A characterization of 3iet words over the alphabet $\{A, B, C\}$ by morphic images of Sturmian words over $\{0, 1\}$ is derived in [5]. Let σ_{01} and σ_{10} be morphisms $\{A, B, C\} \rightarrow \{0, 1\}$ defined by

$$\begin{aligned} \sigma_{01}(A) &= \sigma_{10}(A) = 0, \\ \sigma_{01}(B) &= 01, \quad \sigma_{10}(B) = 10, \\ \sigma_{01}(C) &= \sigma_{10}(C) = 1. \end{aligned}$$

Theorem 9.1 ([5]). *An infinite word $\mathbf{u} \in \{A, B, C\}^{\mathbb{N}}$ is a 3iet word if and only if $\sigma_{01}(\mathbf{u})$ and $\sigma_{10}(\mathbf{u})$ are Sturmian words.*

This theorem was an important tool in [4] for the description of substitutions η from Theorem 8.1 fixing a 3iet word. Since this result is important for our further considerations, we cite it as Theorem 9.5, but first, we need some definitions.

Definition 9.2. Let u and v be finite or infinite words over the alphabet $\{0, 1\}$. We say that u is *amicable* to v , and denote it by $u \propto v$, if there exists a ternary word w over $\{A, B, C\}$ such that $u = \sigma_{01}(w)$ and $v = \sigma_{10}(w)$. In such case, we set $w := \text{ter}(u, v)$ and say that w is the *ternarization* of u and v .

Definition 9.3. Let $\varphi, \psi : \{0, 1\}^* \rightarrow \{0, 1\}^*$ be two morphisms. We say that φ is *amicable* to ψ , and denote it by $\varphi \propto \psi$, if the three following relations hold

$$\begin{aligned}\varphi(0) &\propto \psi(0), \\ \varphi(1) &\propto \psi(1), \\ \varphi(01) &\propto \psi(10).\end{aligned}\tag{19}$$

The morphism $\eta : \{A, B, C\}^* \rightarrow \{A, B, C\}^*$ given by

$$\begin{aligned}\eta(A) &:= \text{ter}(\varphi(0), \psi(0)), \\ \eta(B) &:= \text{ter}(\varphi(01), \psi(10)), \\ \eta(C) &:= \text{ter}(\varphi(1), \psi(1)),\end{aligned}$$

is called the ternarization of φ and ψ and denoted by $\eta := \text{ter}(\varphi, \psi)$.

Remark 9.4. If u and v is a pair of amicable words over $\{0, 1\}$, then $|u|_0 = |v|_0$ and $|u|_1 = |v|_1$. Consequently, if φ and ψ are two amicable morphisms, then they have the same incidence matrix.

Theorem 9.5 ([4]). *Let η be a primitive substitution from Theorem 8.1 fixing a non-degenerate 3iet word \mathbf{u} . There exist Sturmian morphisms φ and ψ having fixed points such that $\varphi \propto \psi$ and $\eta = \text{ter}(\varphi, \psi)$. On the other hand, if φ and ψ are Sturmian morphisms with fixed points such that $\varphi \propto \psi$, then the morphism $\eta = \text{ter}(\varphi, \psi)$ has a 3iet fixed point.*

Example 9.6. Consider the following Sturmian morphisms $\varphi, \psi : \{0, 1\}^* \rightarrow \{0, 1\}^*$,

$$\begin{aligned}\varphi(0) &= 0110101 & \psi(0) &= 1010101 \\ \varphi(1) &= 01101 & \psi(1) &= 10101\end{aligned}$$

We verify the condition given in (19) and in the same time construct the ternarization $\eta = \text{ter}(\varphi, \psi)$. We check that $\varphi(0) \propto \psi(0)$,

$$\begin{aligned}\varphi(0) &= \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline \end{array}, \\ \psi(0) &= \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 1 & 0 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline \end{array}, \\ \eta(A) &= \begin{array}{c} B \quad C \quad A \quad C \quad A \quad C \end{array}\end{aligned}$$

and $\varphi(1) \propto \psi(1)$

$$\begin{aligned}\varphi(1) &= \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline \end{array}, \\ \psi(1) &= \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 1 & 0 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline \end{array}, \\ \eta(C) &= \begin{array}{c} B \quad C \quad A \quad C \end{array}\end{aligned}$$

and lastly that $\varphi(01) \propto \psi(10)$

$$\begin{array}{lcl} \varphi(01) = & \boxed{0\ 1} & \boxed{1} \quad \boxed{0} \quad \boxed{1} \quad \boxed{0\ 1} \quad \boxed{0\ 1} \quad \boxed{1} \quad \boxed{0} \quad \boxed{1} \\ \psi(10) = & \boxed{1\ 0} & \boxed{1} \quad \boxed{0} \quad \boxed{1} \quad \boxed{1\ 0} \quad \boxed{1\ 0} \quad \boxed{1} \quad \boxed{0} \quad \boxed{1} \\ \eta(B) = & B & C \quad A \quad C \quad B \quad B \quad C \quad A \quad C \end{array}$$

We obtained a ternarization of a pair of amicable Sturmian morphisms. By Theorem 9.5, η fixes a 3iet word.

Theorem 9.5 expresses the relation between substitutions fixing 3iet words and Sturmian morphisms. Recall that by a result of [26], all Sturmian morphisms with the same incidence matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be ordered by the relation \triangleleft of conjugation into a chain

$$\xi_1 \triangleleft \xi_2 \triangleleft \cdots \triangleleft \xi_N, \quad \text{where } N = a + b + c + d - 1. \quad (20)$$

This implies that for every i and j such that $1 \leq i < j \leq N$, there exists a word $u \in \{0, 1\}^*$ of length $j - i$ such that $u\xi_i(a) = \xi_j(a)u$ for $a \in \{0, 1\}$.

Lemma 9.7. *Let $\eta = \text{ter}(\varphi, \psi)$, $\eta' = \text{ter}(\varphi', \psi')$, where φ, ψ and φ', ψ' are pairs of amicable Sturmian morphisms over the alphabet $\{0, 1\}$. If $\eta \triangleleft \eta'$, then $\varphi \triangleleft \varphi'$ and $\psi \triangleleft \psi'$, and, moreover, $\varphi = \xi_i$, $\psi = \xi_j$, $\varphi' = \xi_{i'}$, $\psi' = \xi_{j'}$ where $j - i = j' - i'$.*

Proof. Since $\eta \triangleleft \eta'$, there exists a word $w \in \{A, B, C\}^*$ such that $w\eta_1(X) = \eta_2(X)w$ for every $X \in \{A, B, C\}$. We will show that then there exists an amicable pair of words $u, v \in \{0, 1\}^*$ with $|u| = |v|$ such that $w = \text{ter}(u, v)$ and

$$\begin{aligned} u\varphi(b) &= \varphi'(b)u \\ v\psi(b) &= \psi'(b)v \end{aligned} \quad \text{for } b \in \{0, 1\}. \quad (21)$$

It suffices to prove this statement for w of length 1, i.e., $\text{Fst}(\eta'(X)) = w$ for $X \in \{A, B, C\}$. If $w = A$, then necessarily $\text{Fst}(\varphi'(b)) = \text{Fst}(\psi'(b)) = \text{Lst}(\varphi(b)) = \text{Lst}(\psi(b)) = 0$ for $b \in \{0, 1\}$. Therefore $u = v = 0$. If $w = C$, then similarly, $\text{Fst}(\varphi'(b)) = \text{Fst}(\psi'(b)) = \text{Lst}(\varphi(b)) = \text{Lst}(\psi(b)) = 1$ for $b \in \{0, 1\}$, and thus $u = v = 1$. If $w = B$, then $\varphi'(0)$ and $\varphi'(1)$ have prefix 01, and $\psi'(0)$ and $\psi'(1)$ have prefix 10. Thus $u = 01$, $v = 10$ and clearly, $w = \text{ter}(u, v)$.

Now φ and ψ are amicable Sturmian morphisms with the same incidence matrix M , and since φ', ψ' are their conjugates, they also have the same incidence matrix, thus $\varphi = \xi_i$, $\psi = \xi_j$, $\varphi' = \xi_{i'}$, $\psi' = \xi_{j'}$ for some $1 \leq i, j, i', j' \leq N$. The relation $|u| = j - i = j' - i'$ follows from (21). \square

Lemma 9.8. *Let η be a primitive substitution given by Theorem 8.1 fixing a 3iet word. We have*

$$\begin{aligned} & \left(\text{Fst}(\eta_L(A)), \text{Fst}(\eta_L(B)), \text{Fst}(\eta_L(C)) \right) = \\ & \left(\text{Lst}(\eta_R(A)), \text{Lst}(\eta_R(B)), \text{Lst}(\eta_R(C)) \right) = (A, B, B), \end{aligned}$$

or

$$\begin{aligned} & \left(\text{Fst}(\eta_L(A)), \text{Fst}(\eta_L(B)), \text{Fst}(\eta_L(C)) \right) = \\ & \left(\text{Lst}(\eta_R(A)), \text{Lst}(\eta_R(B)), \text{Lst}(\eta_R(C)) \right) = (B, B, C). \end{aligned}$$

Proof. Since the words $\eta(A)$, $\eta(B)$ and $\eta(C)$ are I -itineraries for some interval I , the first letters of $\eta(A)$, $\eta(B)$ and $\eta(C)$ cannot all be distinct. On the contrary, suppose that the discontinuity points α and β of the transformation T belong to the interval I . It implies that these points coincide with the discontinuity points \mathfrak{D} and \mathfrak{C} of the induced map T_I . But this means that T_I is not homothetic to T , which is a contradiction.

By Remark 8.6, the only possibilities for the triple of letters

$$\left(\text{Fst}(\eta(A)), \text{Fst}(\eta(B)), \text{Fst}(\eta(C))\right) \quad \text{and} \quad \left(\text{Lst}(\eta(A)), \text{Lst}(\eta(B)), \text{Lst}(\eta(C))\right)$$

are (A, A, B) , (A, B, B) , (B, B, C) , and (B, C, C) .

We will prove the following claim: Let φ, ψ be the pair of amicable Sturmian morphisms over the alphabet $\{0, 1\}$ such that $\eta = \text{ter}(\varphi, \psi)$.

(i) If $\eta = \eta_L$, i.e., η is the leftmost conjugate of itself, then either

$$\begin{aligned} \psi &= \psi_L \quad \text{and} \quad \left(\text{Fst}(\eta_L(A)), \text{Fst}(\eta_L(B)), \text{Fst}(\eta_L(C))\right) = (A, B, B), \quad \text{or} \\ \varphi &= \varphi_L \quad \text{and} \quad \left(\text{Fst}(\eta_L(A)), \text{Fst}(\eta_L(B)), \text{Fst}(\eta_L(C))\right) = (B, B, C). \end{aligned}$$

(ii) If $\eta = \eta_R$, i.e., η is the rightmost conjugate of itself, then either

$$\begin{aligned} \varphi &= \varphi_R \quad \text{and} \quad \left(\text{Lst}(\eta_L(A)), \text{Lst}(\eta_L(B)), \text{Lst}(\eta_L(C))\right) = (A, B, B), \quad \text{or} \\ \psi &= \psi_R \quad \text{and} \quad \left(\text{Lst}(\eta_L(A)), \text{Lst}(\eta_L(B)), \text{Lst}(\eta_L(C))\right) = (B, B, C). \end{aligned}$$

In order to prove (i), let us discuss the case $\eta = \eta_L$ and $\text{Fst}(\eta(A)) = A$, $\text{Fst}(\eta(C)) = B$. Since $\eta(A) = \text{ter}(\varphi(0), \psi(0))$, necessarily $\text{Fst}(\varphi(0)) = \text{Fst}(\psi(0)) = 0$. As, $\eta(C) = \text{ter}(\varphi(1), \psi(1))$, necessarily $\text{Fst}(\varphi(1)) = 0$ and $\text{Fst}(\psi(1)) = 1$. Thus, the first letter of $\eta(B) = \text{ter}(\varphi(01), \psi(10))$ is B . Therefore the triple (A, A, B) is excluded. Moreover, we see that $\psi = \psi_L$. By the same reasoning, we proceed in the case that $\text{Fst}(\eta(A)) = B$, $\text{Fst}(\eta(C)) = C$ to exclude the triple (B, C, C) and prove $\varphi = \varphi_L$. The proof of (ii), i.e., the case $\eta = \eta_R$ is analogous.

Consider $\left(\text{Fst}(\eta_L(A)), \text{Fst}(\eta_L(B)), \text{Fst}(\eta_L(C))\right) = (A, B, B)$ and $\psi = \psi_L$. If $\xi_1 \triangleleft \dots \triangleleft \xi_N$ are Sturmian morphisms of (20) with the same incidence matrix, then we have $\psi = \xi_1$, and $\varphi = \xi_j$ for some $1 < j \leq N$. Consider now the substitution η_R and denote φ', ψ' the amicable Sturmian morphisms such that $\eta_R = \text{ter}(\varphi', \psi')$. By item (ii), either φ' or ψ' is equal to ξ_N . Due to Lemma 9.7, we know that $\varphi' = \xi_N$, whence by item (ii), the substitution η_R satisfies $\left(\text{Lst}(\eta_R(A)), \text{Lst}(\eta_R(B)), \text{Lst}(\eta_R(C))\right) = (A, B, B)$.

The case $\left(\text{Fst}(\eta_L(A)), \text{Fst}(\eta_L(B)), \text{Fst}(\eta_L(C))\right) = (B, B, C)$ is treated similarly. \square

Corollary 9.9. *Let η be a primitive substitution given by Theorem 8.1 fixing a 3iet word \mathbf{u}_ρ . If η satisfies $\left(\text{Fst}(\eta(A)), \text{Fst}(\eta(B)), \text{Fst}(\eta(C))\right) = (A, B, B)$, then $\rho = \alpha$, and if it satisfies $\left(\text{Fst}(\eta(A)), \text{Fst}(\eta(B)), \text{Fst}(\eta(C))\right) = (B, B, C)$, then $\rho = \beta$.*

Proof. Let I be the interval corresponding to η such that T_I is homothetic to T . Denote $I_X = \{x \in I : R_I(x) = X\}$. If $\left(\text{Fst}(\eta(A)), \text{Fst}(\eta(B)), \text{Fst}(\eta(C))\right) = (A, B, B)$, then the boundary between intervals I_A and I_B , i.e., the discontinuity point of T_I , is equal to the point α . Since T_I is homothetic to T , the homothety map Φ maps the discontinuity points of T to the discontinuity points of T_I , i.e., $\Phi(\alpha) = \alpha$. Since the fixed point of the homothety is equal to the intercept of the infinite word coded by η , we have $\rho = \alpha$. The second implication is analogous. \square

As a byproduct of our results, it is possible, for a given substitution ξ admitting a non-degenerate 3iet word \mathbf{u} as a fixed point, to give a formula allowing to determine the parameters of \mathbf{u} , i.e., the parameters α, β of the transformation T and the intercept ρ such that $\mathbf{u} = \mathbf{u}_\rho$ is a coding of ρ under T . Similar formula for Sturmian morphisms, i.e., those having some word coding an exchange of two intervals as a fixed point, has been given in [21].

The identification of the parameters α and β of the 3iet T is a straightforward task: The values α , $\beta - \alpha$, and $1 - \beta$ are frequencies of the letters A , B and C , respectively, in any infinite word coding some orbit of T . Moreover, the frequencies of letters of a fixed point of a primitive substitution form can be easily determined from the eigenvector corresponding to the dominant eigenvalue of the incidence matrix of the substitution, see [22].

Therefore, the only nontrivial task is to determine the intercept ρ . For this purpose we use the substitution η assigned to the substitution ξ by Theorem 8.1 and its leftmost conjugate η_L . The substitution η has exactly one eigenvalue belonging to the interval $(0, 1)$, see comments below Theorem 8.1. Let this eigenvalue be denoted by λ . Let w be the word of conjugacy between η and η_L , i.e., $\eta(a)w = w\eta_L(a)$ for any $a \in \{A, B, C\}$. Recall that symbols $|w|_A$, $|w|_B$, and $|w|_C$ stand for the number of the letters A , B , and C occuring in w .

Theorem 9.10. *Let $\xi : \{A, B, C\}^* \rightarrow \{A, B, C\}^*$ be a primitive substitution such that it has a fixed point \mathbf{u} . Suppose that \mathbf{u} is a coding of an orbit of a point, say ρ , under a non-degenerate 3iet T with parameters α and β . Let λ , η , η_L and w be as above. We have*

$$\rho = \rho_L + \frac{1}{1 - \lambda}(1 - \alpha, 1 - \alpha - \beta, -\beta) \begin{pmatrix} |w|_A \\ |w|_B \\ |w|_C \end{pmatrix},$$

where $\rho_L = \alpha$ if $\eta_L(A)$ starts with A and $\rho_L = \beta$ if $\eta_L(A)$ starts with B .

Proof. According to Lemma 9.8, $\eta_L(A)$ starts in A or B . Denote by ρ_L the intercept of the 3iet word fixed by η_L . By Corollary 9.9, ρ_L is equal to α if $\eta_L(A)$ starts in A and it is equal to β if $\eta_L(A)$ starts in B . We can use Proposition 8.3 to derive

$$(1 - \lambda)\rho_L = T^n((1 - \lambda)\rho), \quad \text{where } n = |w|.$$

The definition of the transformation T implies the following observation: If w of length n is a prefix of \mathbf{u}_x , then $\mathbf{u}_x = w\mathbf{u}_{T^n(x)}$ and

$$T^n(x) = x + (1 - \alpha, 1 - \alpha - \beta, -\beta) \begin{pmatrix} |w|_A \\ |w|_B \\ |w|_C \end{pmatrix}.$$

Combining these two facts with Remark 8.4, we get

$$(1 - \lambda)\rho = (1 - \lambda)\rho_L + (1 - \alpha, 1 - \alpha - \beta, -\beta) \begin{pmatrix} |w|_A \\ |w|_B \\ |w|_C \end{pmatrix}.$$

The statement follows. □

10 Applications

10.1 Class P conjecture for non-degenerate 3iet

This subsection is devoted to a question coming from another field, namely mathematical physics, where notions from combinatorics on words appear naturally in the study of the spectra of

Schrödinger operators associated to infinite sequences. The question is stated in an article of Hof, Knill and Simon [13] and concerns infinite sequences generated by a substitution over a finite alphabet. The authors show in their paper that if a sequence contains infinitely many palindromic factors (such sequences are called *palindromic*), then the associated operator has a purely singular continuous spectrum. In the same paper, the following class of substitutions is defined.

Definition 10.1. Let φ be a substitution over an alphabet \mathcal{A} . We say that φ belongs to the *class* P if there exists a palindrome p such that for every $a \in \mathcal{A}$ one has $\varphi(a) = pp_a$ where p_a is a palindrome. We say that φ is of *class* P' if it is conjugate to some morphism in class P .

Hof, Knill and Simon ask the following question: “Are there (minimal) sequences containing arbitrarily long palindromes that arise from substitutions none of which belongs to class P ?” A discussion on how to transform this question into a mathematical formalism can be found in [15].

The first result concerning class P was given by Tan in [29]. The author extended class P by morphisms conjugated to the elements of class P , since it is well-known that fixed points of conjugated morphisms have the same set of factors. This extended class is denoted by P' .

The conjecture, stemming from the question of Hof, Knill and Simon, states that every pure morphic (uniformly recurrent) palindromic sequence is a fixed point of a morphism of class P' . It is referred to as ‘class P conjecture’.

In [29], it is shown that if a fixed point of a primitive substitution φ over a binary alphabet is palindromic, then the substitution φ or φ^2 belongs to class P' . In [16], Labbé shows that the assumption of a binary alphabet in the theorem of Tan is essential. He shows that the fixed point of the substitution

$$A \mapsto ABA, B \mapsto C, C \mapsto BAC,$$

is palindromic. The substitution clearly does not belong to class P' . Moreover, no other substitution fixing the same infinite word belongs to class P' . It is easy to see that Labbé’s substitution fixes a degenerate 3iet word, namely 3iet word coding the orbit of $\rho = \frac{2-\sqrt{2}}{4}$ under the 3iet with parameters $\alpha = \frac{1}{2}$ and $\beta = \frac{3-\sqrt{2}}{2}$.

We show that a ternary analogue of the theorem of Tan holds in the context of codings of non-degenerate 3iet with the permutation (321). The following lemma is a generalization of a result obtained for binary alphabets by Tan [29], also shown in [15]. We provide a different proof.

Proposition 10.2. *Let $\varphi : \mathcal{A} \rightarrow \mathcal{A}$ be a non-erasing morphism. The morphism φ is conjugate to $\overline{\varphi}$ if and only if φ is of class P' .*

Proof. (\Leftarrow): Since φ is of class P' , there exists a morphism φ' of class P which is conjugate to φ , i.e., there exists a word w such that $w\varphi(a) = \varphi'(a)w$ or $\varphi(a)w = w\varphi'(a)$ for every letter a .

We can suppose that $w\varphi(a) = \varphi'(a)w$ for every letter a as the other case is analogous. It implies $\varphi(a) = w^{-1}pp_a w$ for some palindromes p_a and \underline{p} . Thus, $\overline{\varphi(a)} = \overline{w}p_a p(\overline{w})^{-1}$ for every letter a . In other words, the morphism $\overline{\varphi}$ is conjugate to $\overline{\varphi'}$. Since $\overline{\varphi'}$ is clearly conjugate to φ , we conclude that φ is conjugate to $\overline{\varphi}$.

(\Rightarrow): Since φ is conjugate to $\overline{\varphi}$, there exists a word $w \in \mathcal{B}^*$ such that for every $a \in \mathcal{A}$, we have

$$\varphi(a)w = \overline{w\overline{\varphi(a)}} \quad \text{or} \quad w\varphi(a) = \overline{\varphi(a)}w.$$

Suppose first that $\varphi(a)w = \overline{w\overline{\varphi(a)}}$ holds. By Lemma 1 in [8], this implies that w is a palindrome. Let $u \in \mathcal{A}^*$ and $c \in \{\varepsilon\} \cup \mathcal{A}$ be such that $w = uc\overline{u}$. We can thus write

$$\varphi(a)uc\overline{u} = uc\overline{u\overline{\varphi(a)}}.$$

By applying $(uc)^{-1}$ from the left and $(c\bar{u})^{-1}$ from the right, we obtain for any $a \in \mathcal{A}$

$$c^{-1}u^{-1}\varphi(a)u = \overline{u\varphi(a)u}^{-1}c^{-1} = \overline{c^{-1}u^{-1}\varphi(a)u}.$$

This means that the word $p_a := c^{-1}u^{-1}\varphi(a)u$ is a palindrome. Set $p := c$. Denote by φ' the morphism defined for all $a \in \mathcal{A}$ by $\varphi'(a) = pp_a = u^{-1}\varphi(a)u$. Obviously, φ is conjugate to φ' which is of class P . Therefore $\varphi \in P'$.

The case $w\varphi(a) = \overline{\varphi(a)}w$ is analogous. \square

We are now in position to complete the proof the main theorem.

Theorem 10.3. *If ξ is a primitive substitution fixing a non-degenerate 3iet word, then ξ or ξ^2 belongs to class P' .*

Proof. Denote by $\eta \in \{\xi, \xi^2\}$ the substitution from Theorem 8.1. There exist intervals I_L and $I_R \subset [0, 1)$ such that $\eta_L(A), \eta_L(B)$ and $\eta_L(C)$ are the I_L -itineraries, $\eta_R(A), \eta_R(B)$ and $\eta_R(C)$ are the I_R -itineraries, and such that T_{I_L} and T_{I_R} are 3iets homothetic to T .

Lemma 9.8 implies that

$$\left(\text{Fst}(\eta_L(A)), \text{Fst}(\eta_L(B)), \text{Fst}(\eta_L(C)) \right) = \left(\text{Lst}(\eta_R(A)), \text{Lst}(\eta_R(B)), \text{Lst}(\eta_R(C)) \right)$$

and this triple of letters equals (A, B, B) or (B, B, C) . Suppose it is equal to (A, B, B) . Note that by Corollary 9.9, η_L fixes the infinite word \mathbf{u}_α .

According to Proposition 8.5, the induced transformation $T_{\overline{I_R}}$ is again homothetic to T and the corresponding substitution is $\overline{\eta_R}$. Since it is the mirror substitution to η_R , we have $\left(\text{Fst}(\overline{\eta_R}(A)), \text{Fst}(\overline{\eta_R}(B)), \text{Fst}(\overline{\eta_R}(C)) \right) = (A, B, B)$. By Corollary 9.9, the substitution $\overline{\eta_R}$ also fixes the infinite word \mathbf{u}_α . Since the intervals I_L and $\overline{I_R}$ are of the same length and are homothetic to the interval $[0, 1)$ with the same homothety center α , necessarily $I_L = \overline{I_R}$ and thus $\overline{\eta_R} = \eta_L$. Consequently, η_R is conjugate to its mirror image. We apply Proposition 10.2 to finish the proof.

In case that $\left(\text{Fst}(\eta_L(A)), \text{Fst}(\eta_L(B)), \text{Fst}(\eta_L(C)) \right) = (B, B, C)$, we proceed in a similar way. In this case, the center of the homothety of the intervals $I_L = \overline{I_R}$ and $[0, 1)$ is β . \square

Let us mention that another analogue of Tan's result is already known for marked morphisms. Recall that a substitution ξ over an alphabet \mathcal{A} is called *marked* if its leftmost conjugate ξ_L and its rightmost conjugate ξ_R satisfy

$$\text{Fst}(\xi_L(a)) \neq \text{Fst}(\xi_L(b)) \quad \text{and} \quad \text{Lst}(\xi_R(a)) \neq \text{Lst}(\xi_R(b))$$

for distinct $a, b \in \mathcal{A}$. It can be shown that if ξ is marked, then all its powers are marked. In [15], it is shown that for a marked morphism ξ with fixed point \mathbf{u} having infinitely many palindromes, some power ξ^k belongs to class P' .

Our Lemma 9.8 shows that a substitution fixing a non-degenerated 3iet word cannot be marked. Theorem 10.3 thus provides a new family of substitutions satisfying class P conjecture.

10.2 Properties of 3iet preserving substitutions

A morphism which maps a 3iet word to a 3iet word is called 3iet preserving. Morphisms which preserve 2iet words, i.e. Sturmian words, are called Sturmian and they have been extensively studied for many years. Sturmian morphisms form a monoid which is generated by three morphisms only

[20]. In contrary to Sturmian morphisms, the class of 3iet preserving morphisms is not completely described. Only partial results are known. For example, the monoid of 3iet preserving morphisms is not finitely generated [2] and contains the ternarizations we defined in Section 9, see [4]. Our previous considerations lead to some comments on properties of 3iet preserving substitution.

- Our results of Sections 7 and 8 allow us to say more about the structure of substitutions fixing 3iet words.

Corollary 10.4. *Let η be a primitive substitution of Theorem 8.1 fixing a non-degenerate 3iet word over the alphabet $\{A, B, C\}$. We have*

$$\eta(B) = \omega_{AC \rightarrow B}(\eta(AC)) = \omega_{CA \rightarrow B}(\eta(CA)) \quad \text{or} \quad \eta(B) = \omega_{B \rightarrow CA}(\eta(AC)) = \omega_{B \rightarrow AC}(\eta(CA)).$$

Proof. By Theorem 8.1, η corresponds to an interval I such that T_I is homothetic to T . Since T is non-degenerate, also T_I is non-degenerate, and therefore its discontinuity points $\mathfrak{C}, \mathfrak{D}$ are distinct. By Proposition 7.1, the three I -itineraries are of the form given by cases (i) or (ii). \square

Example 10.5. We can illustrate the above corollary on substitutions from Example 9.6. We have

$$\begin{aligned} \varphi(0) &= 0110101 & \psi(0) &= 1010101 \\ \varphi(1) &= 01101 & \psi(1) &= 10101 \end{aligned} \quad ,$$

and

$$\eta(A) = BCACAC, \quad \eta(B) = BCACBBCAC, \quad \eta(C) = BCAC.$$

We can check that η satisfies the property given in Corollary 10.4, namely that

$$\begin{aligned} \eta(B) = BCACBBCAC &= \omega_{AC \rightarrow B}(\eta(AC)) = \omega_{AC \rightarrow B}(BCACACBCAC) = \\ &= \omega_{CA \rightarrow B}(\eta(CA)) = \omega_{CA \rightarrow B}(BCACBCACAC). \end{aligned}$$

The above corollary implies a relation of numbers of occurrences of letters in letter images of η which may be used to get the following relation:

$$M_\eta \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} |\eta(A)|_A \\ |\eta(A)|_B \\ |\eta(A)|_C \end{pmatrix} - \begin{pmatrix} |\eta(B)|_A \\ |\eta(B)|_B \\ |\eta(B)|_C \end{pmatrix} + \begin{pmatrix} |\eta(C)|_A \\ |\eta(C)|_B \\ |\eta(C)|_C \end{pmatrix} = \pm \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Thus, $(1, -1, 1)^\top$ is an eigenvector of the incidence matrix of η corresponding to the eigenvalue 1 and -1 , respectively. This fact has been already derived in [3] by other methods.

- One can ask whether a substitution η fixing a 3iet word can be in the same time the leftmost and rightmost conjugate of itself, i.e. $\eta = \eta_L = \eta_R$. It can be easily seen that such a situation never occurs for non-degenerate 3iets. Indeed, the proof of Theorem 10.3 implies that for any primitive substitution η fixing a non-degenerate 3iet, η_L and $\overline{\eta_R}$ fix the same infinite word \mathbf{u}_ρ , where $\rho \in \{\alpha, \beta\}$. If, moreover, $\eta_R = \overline{\eta_R}$, then by Proposition 8.5 we have $1 - \rho = \rho$. This implies that $\rho = \frac{1}{2} \in \{\alpha, \beta\}$. However, this cannot happen for a non-degenerate 3iet T .
- It can be observed from Lemma 9.8 that given a substitution η fixing a 3iet word, its leftmost conjugate η_L has always two fixed points, namely either $\lim_{n \rightarrow \infty} \eta_L^n(A)$ and $\lim_{n \rightarrow \infty} \eta_L^n(B)$, or $\lim_{n \rightarrow \infty} \eta_L^n(B)$ and $\lim_{n \rightarrow \infty} \eta_L^n(C)$. One can show that one of these fixed points is a coding of the point $\rho = \alpha$ or β , respectively, under a 3iet T . The other fixed point is a coding of the same point, but under an exchange of three intervals defined over $(0, 1]$, where all the intervals are of the form $(\cdot, \cdot]$.

Example 10.6. Consider the substitution η from Example 9.6. We have

$$\eta_L(A) = ACBCAC, \quad \eta_L(B) = BBCACBCAC, \quad \eta_L(C) = BCAC.$$

This substitution has two fixed points, namely

$$\begin{aligned} \lim_{n \rightarrow \infty} \eta_L^n(A) &= ACBCACBCACBBCACBCACBCACACBCACBCAC \cdots, \\ \lim_{n \rightarrow \infty} \eta_L^n(B) &= BBCACBCACBBCACBCACBCACACBCACBCAC \cdots. \end{aligned}$$

It can be verified that the two infinite words differ only by the prefix AC vs. B . The infinite word \mathbf{u}_α , coding α under the 3iet T is equal to the fixed point $\lim_{n \rightarrow \infty} \eta_L^n(B)$.

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